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Harmonic Analysis and Superconformal Gauge Theories in Three Dimensions from AdS/CFT Correspondence

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Abstract

In this thesis I review various aspects of the AdS_4/CFT_3 correspondence, where AdS_4 supergravity arises from compactification of M–theory on a coset space G/H and preserves $\mathcal{N} < 8$ supersymmetries. One focal point of my review is that the complete spectrum of such $\mathcal{N}$–extended supergravity can be determined by means of harmonic analysis on the homogeneous space G/H. This spectrum can be matched with the candidate conformal theory on the boundary, in this way providing very non–trivial checks of the AdS/CFT correspondence. Furthermore, this spectrum can be useful to study the representation theory of $\mathcal{N}$–extended supersymmetry on AdS_4, namely representation theory for the superalgebra of $Osp(\mathcal{N}|4)$. I review $Osp(\mathcal{N}|4)$ representation theory, and derive the translation vocabulary between states of AdS_4 supergravity and conformal superfields on the boundary, by means of the double interpretation of $Osp(\mathcal{N}|4)$ unitary irreducible representations. In the cases of $\mathcal{N} = 2, 3$, using results from harmonic analysis I give the complete structure of all supermultiplets. Harmonic analysis as a method to determine spectra of supergravity compactifications is explained. Calculations are explicitly performed in the case $G/H = M^{111} = (SU(3) \times SU(2) \times U(1))/(SU(2) \times U(1) \times U(1))$, preserving $\mathcal{N} = 2$ supersymmetries. For this manifold, and also for the case $G/H = Q^{111} = (SU(2) \times SU(2) \times SU(2))/(U(1) \times U(1))$, I describe the construction of a candidate dual superconformal theory on the boundary. This construction is based on geometrical insight provided by the properties of the metric cone $\mathcal{C}(G/H)$ transverse to the $M2$–brane wordvolume.

Contents

Introduction

The AdS/CFT correspondence is a recent interesting development in the context of string theory and M-theory. It is based on the conjecture, proposed by J. Maldacena [1] at the end of '97 and further developed by E. Witten [2] and S. Gubser, I. Klebanov, A. Polyakov [3], that string/M theory on some backgrounds of the form

$$AdS_d \times X_{11-d} \tag{0.0.1}$$

or

$$AdS_d \times X_{10-d}, \tag{0.0.2}$$

where X is a compact space and AdS is the anti–de Sitter space [4], is equivalent to a *superconformal quantum field theory* (SCFT) on the boundary of the anti–de Sitter space. Actually, this boundary coincides with compactified Minkowski space in $d-1$ dimensions. The superconformal theory can also be viewed as a theory defined on a stack of N coincident branes, D–branes in ten dimensions or M–branes in eleven dimensions, with gauge group $U(N)^k$ or $SU(N)^k$. For example [1], the string theory on $AdS_5 \times S^5$ is equivalent to the $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ and living on the four dimensional boundary of AdS_5.

A striking point of this correspondence is that in the limit $g_{YM}^2 N \longrightarrow \infty$, $g_{YM}^2 \longrightarrow 0$ the duality is between *classical supergravity* and a strongly coupled SCFT, since the string/M-theory corrections are of order $1/N$. So it is possible to determine physical observables of the quantum theory on the boundary by classical supergravity calculations on the bulk.

The starting point of this conjecture has been the observation that the isometry group of string/M theory on AdS_d is $SO(2, d-1)$, and this is also the conformal group in $d-1$ dimensions. So, these two theories have the same symmetry. The conjecture, in its more complete formulation, is that there is a one to one correspondence between supergravity fields Φ on the bulk and conformal primary operators $\mathcal{O}$ on the boundary, and the generating functional of the correlators of the boundary theory can be written, in the appropriate limit, in terms of the classical supergravity action.

For every off shell field configuration on the boundary of AdS Φ_0 there is a unique on shell field configuration Φ that is regular on the bulk and which has Φ_0 as boundary value. Then the field Φ on the bulk depends on its value Φ_0 on the boundary, and the generating functional is [2]

$$Z(\Phi_0) \equiv \left\langle e^{\int_{AdS_{d-1}} \Phi_0 \mathcal{O}} \right\rangle \overset{AdS/CFT}{=} e^{-S(\Phi(\Phi_0))}, \tag{0.0.3}$$

where, in the appropriate limit, $S(\Phi_0)$ is the classical supergravity action on the bulk.

We can view the correspondence as between the states of supergravity compactified on the bulk, which are the states created by the on–shell fields Φ, and conformal operators on the boundary. In particular, the BPS states of compactified supergravity correspond to short primary superconformal operators of the boundary theory, which are protected against quantum corrections.

Another feature of this correspondence is that the set of the Kaluza Klein states on the bulk must completely match with the set of conformal operators on the boundary; the energies of these Kaluza Klein states correspond to the conformal weights of the operators on the boundary; in particular, in the case of BPS states, this correspondence can be checked without calculations at the quantum level, since the conformal weights are protected against quantum corrections. This is true also for the other quantum numbers.

The most studied cases of AdS/CFT correspondence are those with $D = 10,\ d = 5$, where the bulk theory is the compactification of string theory or, at low energy, ten dimensional supergravity, on

$$AdS_5 \times X_5. \tag{0.0.4}$$

This case has become very popular since the bulk theory is string theory which is well known, and the boundary theory has four dimensions. In the present thesis I rather consider the case $D = 11,\ d = 4$. Then the bulk theory is the compactification of $M-$theory, or, at low energy, eleven dimensional supergravity, on

$$AdS_4 \times X_7. \tag{0.0.5}$$

There are various reasons of this choice.

- Since a formulation of the quantum theory for the fundamental degrees of freedom of M–theory is lacking, it is of utmost interest to explore the properties of all its vacua.

- The study of AdS_4/CFT_3 correspondence is useful to understand the three dimensional conformal field theories, which are not well known. For example, a classification of the central charges as known for four dimensional CFTs [5] is lacking for three dimensional CFTs. Furthermore, three dimensional conformal field theories are intrinsically interesting, being related to statistical mechanics.

- In the eighties the Kaluza Klein compactifications of eleven dimensional supergravity on $AdS_4 \times X_7$ background was studied in order to find a realistic theory as a supergravity compactification. In particular, the manifold $X_7 = M^{111}$ (that I will introduce in the following) was studied [6],[7],[8], having this manifold isometry $SU(3) \times SU(2) \times U(1)$ as the standard model group. That way resulted to be wrong, because such compactifications yield theories with unphysical cosmological constants and no chiral fermions. Today, in the completely new context of AdS/CFT correspondence, the anti–de Sitter space resulting by these compactifications is no more a flaw of the theory, but an asset, and all those results can be utilized in the new perspective.

In general, there should be AdS/CFT correspondence if $AdS_4 \times X_7$ is a classical supergravity solution. This restricts the possible choices of the compact manifold X_7. At the moment, there are three kinds of $AdS_4 \times X_7$ correspondences that have been studied:

- $X_7 = S^7$, that yields maximal $\mathcal{N}=8$ supergravity [1], [9], [10].
- $X_7 = S^7/\Gamma$ with Γ a discrete group, namely, an orbifold of S^7; these cases yield consistent truncations of $\mathcal{N}=8$ supergravity [9], [11].
- $X_7 = G/H$ coset manifold that is also an Einstein space; these cases yield $\mathcal{N}<8$ supergravities which are not truncations of the $\mathcal{N}=8$ theory, but completely new ones [12] (see [13] for the $AdS_5 \times X_5$ case).

The same considerations hold true for the correspondence with $AdS_5 \times X_5$ or other choices of D, d.

Up to now, several checks have been found of the AdS_4/CFT_3 correspondence. The most complete of them refer to the case $X_7 = S^7$, or to the cases of S^7 orbifolds. In most of these cases, the correspondence of the BPS Kaluza Klein states of supergravity with the BPS superfield operators of the SCFT has been checked. Furthermore, in those cases where the correlators of the SCFT are known in the strong coupling limit, the formula (0.0.3) comes out true. This holds also for other choices of D, d.

Nevertheless, these are not the strongest possible checks of this type, particularly for the check of the spectrum. In the maximal supersymmetric case $X_7 = S^7$, the energies of the Kaluza Klein states and the conformal weights of the superconformal operators depend only on their R–symmetry representations. Being the superisometry $Osp\,(8|4)$ the same, it is not really surprising that the energies and the conformal weights actually coincide. And the truncations of these theories do not contain any really new information. On the contrary, for supergravities on $X_7 = G/H \neq S^7$, these energies and conformal weights depend also on the so called *flavour group* representations, and this dependence is ruled not only by the $Osp\,(\mathcal{N}|4)$ representation theory, but also - on the bulk side - by the geometry of the compactification on X_7. Furthermore, the theories with $\mathcal{N}<8$ are less constrained than the maximal supersymmetric one. Then, a check of the spectrum in a case with $X_7 = G/H \neq S^7$ is more significant than in the maximally supersymmetric case.

This thesis is mainly based on the work done during my last Ph.D. year in the collaborations [14],[15],[16],[17], in order to discuss non–trivial checks of AdS/CFT correspondence in the cases

$$AdS_4 \times \left(\frac{G}{H}\right)_7 \tag{0.0.6}$$

preserving $\mathcal{N}<8$ supersymmetries. It is also my aim to make a systematic review of the algebraic and geometric foundations of this correspondence.

We have studied in detail the case of the manifold

$$X_7 = M^{111} = \frac{SU\,(3) \times SU\,(2) \times U\,(1)}{SU\,(2) \times U\,(1) \times U\,(1)} \tag{0.0.7}$$

preserving $\mathcal{N}=2$ supersymmetries. Furthermore, we have studied, with less detail, the cases of the manifold

$$X_7 = Q^{111} = \frac{SU\,(2) \times SU\,(2) \times SU\,(2)}{U\,(1) \times U\,(1)} \tag{0.0.8}$$

preserving $\mathcal{N}=2$ supersymmetries, and of the manifold

$$X_7 = N^{010} = \frac{SU\,(3)}{U\,(1)} \tag{0.0.9}$$

preserving $\mathcal{N}=3$ supersymmetries.

In [16], we have constructed superconformal theories candidate to be dual to supergravity on $AdS_4 \times M^{111}$ and to supergravity on $AdS_4 \times Q^{111}$ [1]. Matching these theories with the supergravity on the bulk prevoiously derived [14], we found new non–trivial checks of AdS/CFT correspondence. Furthermore, in order to reach these results, other results were obtained as a byproduct [14],[15],[16],[17].

- We built a case of AdS/CFT correspondence following all the path, from the development of the supergravity theory on the bulk to the development of the candidate superconformal field theory on the boundary. This gave us a deeper understanding of the mechanism of AdS/CFT correspondence, expecially on the relations between the conformal superfields on the boundary and the Kaluza Klein spectrum on the bulk [15].

- We used the techniques of harmonic analysis in order to find the complete spectrum of supergravity on $AdS_4 \times M^{111}$ [14]. These techniques had been developed in the eighties [19], [20], [21], [22] but this is the first time the complete spectrum of an intricate case as (0.0.7) has been worked out; now we know more about how to handle such problems. Furthermore, this spectrum has value as a supergravity result, even out of the AdS/CFT correspondence context.

- Up to know, the structure of several $\mathcal{N}=2$ and $\mathcal{N}=3$ AdS_4 supermultiplets was not known. The spectra of supergravities we found give us the lacking information on the general representation theory of $\mathcal{N}=2$ and $\mathcal{N}=3$ supersymmetry on AdS_4, and the complete structure of all $\mathcal{N}=2$ [14] and $\mathcal{N}=3$ [17] supermultiplets[2], completing the results of [23], [24].

An analogous program has been carried out by [13], [25], [26] in the case of $AdS_5 \times X_5$ with $X_5 = T^{11} = \frac{SU(2)\times SU(2)}{U(1)}$. In this case the conformal theory has been found in [13] as a deformation of an orbifold theory with larger supersymmetry. The Kaluza Klein spectrum of the corresponding supergravity has been worked out in [25], and a comparison with superfields on the boundary theory [26] gave another non trivial check of the AdS/CFT correspondence. Furthermore, the same program has recently been carried out in [27] for an other $AdS_4 \times X_7$ case, the one of the Stiefel manifold $X_7 = SO(5)/SO(3)$, finding similar results.

In this thesis I present our results in a systematical and didactic form, sometimes going into details more than the papers [14], [15], [16], [17]. Furthermore, some technical details are new and unpublished.

Contents of the thesis

The present thesis is organized as follows.

In chapter 1 I review some basic concepts of the AdS/CFT correspondence, in order to explain the background and the motivation of the subsequent work. Then, I consider

[1]The construction of the SCFT dual to supergravity on $AdS_4 \times N^{010}$ is in preparation [18].

[2]With the exception of the $\mathcal{N}=3$ short supermultiplets with $J_0 = 1/2$ and $J_0 = 3/2$.

explicitly the case I am interested in, which is

$$AdS_4 \times \left(\frac{G}{H}\right)_7 . \tag{0.0.10}$$

In chapter 2 I study the representation theory of $Osp\,(\mathcal{N}|4)$, that is, of supersymmetric theories on AdS_4. With the help of Lie algebra techniques I show the double interpretation of the $Osp\,(\mathcal{N}|4)$ unitary irreducible representations, as states of bulk supergravity and as superfields of a boundary superconformal theory. In the $\mathcal{N}=2$ case I explicitly explain how to know, given a state on the bulk, which is the corresponding conformal operator on the boundary [15]. Furthermore I give the complete structure of all the $Osp\,(\mathcal{N}|4)$ unitary irreducible representations, namely, the supermultiplets of AdS_4 supersymmetry, in the cases $\mathcal{N}=2$ [14] and $\mathcal{N}=3$ [17]. I explain how to find this structure by the matching of results found with the Freedmann Nicolai method of norms [24] with results given by Kaluza Klein spectra.

In chapter 3 I explicitly derive the complete spectrum of supergravity compactified on

$$AdS_4 \times M^{111} \tag{0.0.11}$$

using harmonic analysis [14]. This is a very powerful method, which allow to solve a differential equation problem by purely algebraic calculations. A detailed description of all the mathematical tools used and of our derivation is given.

In chapter 4 I build candidate SCFT's dual by AdS/CFT correspondence to supergravity compactified on $AdS_4 \times M^{111}$ and to supergravity compactified on $AdS_4 \times Q^{111}$ [16]. Unfortunately, while when the compact manifold is the seven sphere or an orbifold of the seven sphere there is a straightforward way to build the conformal theory, by relating it to a ten dimensional string theory with D–branes, this seems not to be possible when the compact manifold is a coset manifold as M^{111} or Q^{111}. One have to use geometrical intuition to argue the fundamental field content and the gauge group of the theory. However, I show that having these theories a toric description, there are strong arguments to argue them, and the results fit surprisingly with the bulk theory.

Chapter 1

AdS/CFT Correspondence and G/H $M-$branes

1.1 The AdS/CFT Correspondence

In this section I review some basic concepts of the AdS/CFT correspondence, in order to explain the background and the motivation of the subsequent work. Several excellent and complete reviews on this wide field of research are available in the literature [28], [29].

1.1.1 The Maldacena Conjecture for $AdS_5 \times S^5$

Let us consider IIB string theory on flat ten dimensional Minkowski space, with N coincident $D3$–branes. The perturbative excitations of this theory are the closed strings, which are the excitations of Minkowski empty space, and the open strings, which can end only on the D–branes, and are the excitations of the D–branes themselves.

Let us consider the low energy limit of the system, namely, take into account only the energies lower than the string scale $1/\sqrt{\alpha'}$

$$E\sqrt{\alpha'} \ll 1 \tag{1.1.1}$$

keeping all the dimensionless parameters (g_s, N) fixed. Then only the massless string states can be excited. The effective action of massless modes, obtained by integrating out the massive fields, has the form

$$S = S_{\text{bulk}} + S_{\text{brane}} + S_{\text{int}}. \tag{1.1.2}$$

- S_{bulk} is the action of ten dimensional supergravity; in the low energy limit [1] it becomes the action of free ten dimensional supergravity in Minkowski space.

- S_{brane} is defined on the $3+1$–dimensional brane worldvolume, and in the low energy limit becomes the action of $\mathcal{N}=4$ super Yang Mills (SYM) theory with gauge group $U(N)$, and
$$g^2_{YM} = 4\pi g_s. \tag{1.1.3}$$
Notice that this is a superconformal field theory (SCFT).

[1] In the actual calculations, the simplest way to perform the low energy limit (1.1.1) is to send $\alpha' \longrightarrow 0$; however we must remind that if one wants to be rigorous, only dimensionless quantities can be sent to zero; $\alpha' \longrightarrow 0$ is a shorthand notation for $E\sqrt{\alpha'} \ll 0$.

- S_{int} describes the interaction between the brane and the bulk, and in the low energy limit disappears.

Then in the low energy limit there are two decoupled systems, free IIB supergravity on Minkowski space and $\mathcal{N}=4$ SYM theory with gauge group $U(N)$.

But string theory can be also viewed from the so–called macroscopic point of view. The absorption of closed strings by the D–branes can be seen also as the interaction of the string modes with a non–trivial supergravity background, due to a massive and charged source localized at the position of the D–branes. In other words, the D–branes behave as massive and charged objects, sources of the supergravity fields. The IIB supergravity background is the following p–brane solution:

$$\begin{aligned}
ds^2 &= f^{-1/2}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+f^{1/2}\left(dr^2+r^2d\Omega_5^2\right)\\
F_5 &= (1+*)\,dtdx_1dx_2dx_3df^{-1}\\
f &= 1+\frac{R^4}{r^4}, \qquad R^4\equiv 4\pi g_sN\alpha'^2,
\end{aligned} \tag{1.1.4}$$

where both the mass and the five–form charge (per unit volume) are proportional to the number of the branes N. Furthermore this solution is a BPS solution, namely, it preserves half of the IIB supersymmetry.

This is a black brane solution, with an horizon at $r=0$. The energy E_p of an object as measured by an observer at a constant position r and the energy E measured by an observer at infinity are related by the redshift factor

$$E=f^{-1/4}E_p, \tag{1.1.5}$$

then as an object is brought near the horizon, it appears with lower energy to the observer at infinity.

We want to consider only the *low energy IIB* string theory excitations on this background, where I mean low energy from the point of view of an observer at infinity. There are two kinds of low energy excitations: the massless particles propagating in the bulk region (that is, $r/R\gg 1$) with large wavelength, and any kind of excitation if it is close enough to the horizon. These two kinds of excitations are decoupled. The bulk massless particles cannot excite the near horizon region, because the cross section $\sigma\sim R^8\omega^3$ is small in the low energy limit (corresponding to big particle wavelengths); we can reformulate this phenomenom saying that the particles cannot be absorbed in this limit because their wavelengths are bigger than the typical gravitational size of the brane. On the other hand, the near horizon excitations have to climb an high potential barrier to escape from the asymptotic region.

In the near horizon region, defined by $r\ll R$, $f\sim R^4/r^4$, so the metric becomes

$$ds^2=\frac{r^2}{R^2}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+R^2\frac{dr^2}{r^2}+R^2d\Omega_5^2\,, \tag{1.1.6}$$

that is the metric of

$$AdS_5\times S^5 \tag{1.1.7}$$

with $R=\left(4\pi g_sN\alpha'^2\right)^{1/4}$ curvature radius of AdS_5 and of S^5. From this point of view, N is the flux of the five–form through S^5.

To be more precise about the near horizon limit, a string excitation has $E_p \sim \frac{1}{\sqrt{\alpha'}}$; an observer at infinity sees the energy

$$E = f^{-1/4} E_p \sim \frac{r}{\sqrt{\alpha'}} E_p \sim \frac{r}{\alpha'} \,. \tag{1.1.8}$$

In the low energy limit $E\sqrt{\alpha'} \ll 1$, then,

$$\frac{r}{\sqrt{\alpha'}} \ll 1 \,. \tag{1.1.9}$$

This means that any excitation of string theory does survive if it is enough close to the horizon to satisfy (1.1.9). We can express this by a coordinate redefinition:

$$U \equiv \frac{r}{\sqrt{4\pi g_s N \alpha'}} = \frac{r}{R^2} \,. \tag{1.1.10}$$

In terms of U, the energies of the near horizon string excitations are finite, and the metric (1.1.6) becomes

$$\begin{aligned} ds^2 &= \sqrt{4\pi g_s N}\alpha' \left[U^2 \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \frac{dU^2}{U^2} + d\Omega_5^2 \right] \\ &= R^2 \left[U^2 \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right) + \frac{dU^2}{U^2} + d\Omega_5^2 \right] \,. \end{aligned} \tag{1.1.11}$$

We have derived this metric as a near horizon geometry, that is, the (1.1.11) is the metric in the region $U \ll 1/R$. Here R is only a constant factor in front of the metric. We can rescale the coordinates, so that the region $U \ll 1/R$ describes an entire AdS_5 space. In other words, we blow up the near horizon (or "throat") region of the (1.1.4) geometry into the entire $AdS_5 \times S^5$ space. The supergravity excitations of the throat coincide with the excitations of $AdS_5 \times S^5$ supergravity.

The low energy theory, then, consists on these two decoupled parts, the free IIB supergravity on Minkowski space and, near the horizon, the $AdS_5 \times S^5$ IIB string theory (with all the excitations).

From both the points of view, then, in the low energy limit there are two decoupled systems, one of which is the free empty IIB supergravity. This suggests that the second systems appearing in both description may be dual, namely, mathematically equivalent.

This is the **Maldacena conjecture**: the $\mathcal{N} = 4$ $U(N)$ SYM quantum field theory in $3+1$ dimensions is dual to IIB string theory on $AdS_5 \times S^5$ background.

In the above reasoning we have kept fixed the two dimensionless parameters of the theory, g_s and N, or, equivalently, g_s and $X \equiv 4\pi N g_s$ (which is the t'Hooft coupling). Let us consider various limits of these parameters.

- When, as in the above reasoning,

$$X,\; g_s \;\; \text{finite}\,, \tag{1.1.12}$$

we have the correspondence between *IIB string theory on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ four dimensional SYM theory with gauge group $U(N)$, N finite and t'Hooft coupling finite.*

- When

$$\begin{aligned} X & \quad \text{finite} \\ g_s & \longrightarrow 0 \\ N & \longrightarrow \infty\,, \end{aligned} \tag{1.1.13}$$

the above reasoning does not change, because g_s, N appear only in the combination X; in particular, $\sqrt{\alpha'} \sim R$ remains true.

The correspondence is between classical *IIB string theory (that is, with only tree diagrams, because $g_s \longrightarrow 0$) on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ four dimensional SYM theory with gauge group $U(N)$, $N \longrightarrow \infty$, X finite.*

- When

$$\begin{aligned} X & \longrightarrow \infty\,, \\ g_s & \longrightarrow 0 \\ N & \longrightarrow \infty\,, \end{aligned} \tag{1.1.14}$$

we have

$$R = X^{1/4}\sqrt{\alpha'} \gg \sqrt{\alpha'}\,. \tag{1.1.15}$$

On the bulk, the classical supergravity excitations decouple from the other string excitations. In fact, ten dimensional supergravity on the $AdS_5 \times S^5$ background is a theory whose dynamical fields are the fluctuations around this background. These fields can be expanded in S^5 harmonics, yielding a tower of five dimensional supergravity Kaluza Klein fields, whose masses are of order $m \sim 1/R$, and whose energies are of order $E_p^{KK} \sim 1/R$. The string excitations, instead, have energies $E_p^s \sim 1/\sqrt{\alpha'}$. The energies as seen by an observer at infinity are then respectively

$$\begin{aligned} E^{KK} & \sim \frac{r}{R^2} \\ E^s & \sim \frac{r}{R\sqrt{\alpha'}} \gg E^{KK}\,. \end{aligned} \tag{1.1.16}$$

So the supergravity (Kaluza Klein) excitations have finite energy in terms of the coordinate $U = r/R^2$, while the string excitations decouple. In other words, the IIB string theory on $AdS_5 \times S^5$ becomes classical supergravity on that background, because the string length is much smaller than the characteristic length of the space, R.

The correspondence is between *classical supergravity on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ four dimensional SYM theory with gauge group $U(N)$, $N \longrightarrow \infty$, $X \longrightarrow \infty$.*

In the following I will consider mainly the last limit (1.1.14), that is the one which has received most confirmations, and then is the most firmly founded version of the correspondence. Notice that in this limit the brane theory is a *strongly coupled* theory, being X large. The AdS/CFT correspondence in the limit (1.1.14), then, relates a weakly coupled theory with a strongly coupled one, in different dimensions.

The argument of the decoupling limit sketched above is not the unique motivation of the Maldacena conjecture. There are a lot of previous results, observations, open problems, that can be better understood in the context of this conjecture.

- The idea that string theories can describe gauge theories dates back to the origin of string theory. In particular, as t'Hooft showed [30] that the large N limit of $SU(N)$ gauge theory is formally similar to perturbative string theory, long efforts have been done to find an exact gauge field/string duality. In this context, it has also been suggested [31] that four dimensional $SU(N)$ Yang Mills theory could be dual to a five dimensional string theory.

- A great advance in non–perturbative string theory has been the discovery [32] that a system of several D–branes in string theory can be described as a black p–brane solution of suypergravity. In particular, this yielded the first microscopic explanation of the Bekenstein Hawking entropy: A. Strominger and C. Vafa [33] considered IIB string theory compactified on a five dimensional compact manifold, and a system of intersecting D–branes wrapped around the compact manifold; they worked out the entropy in both the description, in the 'microscopic' one by counting the D–branes states, in the 'macroscopic' one by applying the Bekenstein Hawking formula, and the two results coincide.

  But in the case of N D–branes on the non–compact space, the results [34] was similar but different:

$$S_{\text{Bekenstein Hawking}} = \frac{\pi^2}{2} N^2 V_3 T^3 \qquad S_{D-\text{branes}} = \frac{2\pi^2}{3} N^2 V_3 T^3 \,. \tag{1.1.17}$$

  This result is meaningful in the context of AdS/CFT correspondence. In fact, the Bekenstein Hawking calculation applies in the supergravity limit, that as I said is the strong coupling limit of the gauge theory on the brane. Conversely, the D–brane calculation is perturbative, then gives the entropy in the weak coupling limit of the gauge theory. The two results, then, differ by the renormalization group flow of a smooth function, that yields the factor $\frac{2}{3}$.

- It has been derived [35] the absorption cross–section of massless bulk excitations from the system of coincident D–branes, in two ways. First, with the D–branes description, looking at the process of closed strings that become open strings on the branes. Second, with the supergravity description (1.1.4); as I said, there is a potential barrier separating the bulk from the near horizon geometry, so waves incident from $r \gg R$ penetrate into the near horizon geometry with a certain cross–section.

  These two cross sections coincide:

$$\sigma = \frac{g_s^2 \alpha'^4 \omega^3 N^2}{32\pi}. \tag{1.1.18}$$

  The meaning of this result is clear in the context of AdS/CFT correspondence. In the D–brane description, a particle incident from the asymptotic infinity is converted into an excitation of the stack of D–branes, namely, into an excitation of the gauge theory on the world volume. In the supergravity description, a particle incident from the asymptotic region tunnels into the $r \ll R$ region and produces an excitation of the near horizon geometry. These two descriptions of the absorption process give the same cross–sections because the excitations of $AdS_5 \times S^5$ supergravity correspond to the excitations of the $\mathcal{N} = 4$ SYM theory.

But the key to the Maldacena conjecture has been a crucial observation on the symmetry groups. The isometry group of AdS_5 space is $SO(4,2)$, but this is also the conformal group in four dimensions. Furthermore, the isometry of the compact space S^5 is $SO(6) = SU(4)$, and this is also the R–symmetry (namely, the automorphism group of the superalgebra) of $\mathcal{N} = 4$ SYM theory. More generally, the isometry of the supergravity background $AdS_5 \times S^5$ is the supergroup $SU(2,2|4)$, that is also the superconformal symmetry of the $\mathcal{N} = 4$ SYM theory. The fact that these theories have the same symmetry is the first hint that they could be equivalent, although they live in different dimensions.

1.1.2 The Maldacena Conjecture for $AdS_4 \times S^7$

The Maldacena conjecture can be extended to other cases. Let us consider M–theory on flat eleven dimensional Minkowski space, with N coincident $M2$–branes. In this case, instead of the string length $\sqrt{\alpha'}$ there is the Planck length l_p, and there is no parameter analogous to the string coupling g_s; the only dimensionless parameter is N. Let us consider the low energy limit,

$$E l_p \ll 1\,, \tag{1.1.19}$$

with N fixed.

It is not known which are the perturbative excitations of this theory, because the quantum M–theory has not been found yet, however it is known that the low energy excitations on the bulk are described by eleven dimensional supergravity, and that it is possible to define superconformal field theories on the worldvolume of the $M2$–branes.

From the macroscopic point of view, the $M2$–branes behave as massive and charged objects, sources of a supergravity background of the form

$$\begin{aligned} ds^2 &= f^{-2/3}\left(-dt^2 + dx_1^2 + dx_2^2\right) + f^{1/3}\left(dr^2 + r^2 d\Omega_7^2\right) \\ F_5 &= dt dx_1 dx_2 df^{-1} \qquad (1.1.20) \\ f &= 1 + \frac{R^6}{r^6}, \qquad R^6 \equiv 32\pi^2 N l_p^6. \qquad (1.1.21) \end{aligned}$$

In the near horizon region $r \ll R$, $f \simeq R^6/r^6$, so the metric becomes

$$ds^2 = \frac{r^4}{R^4}\left(-dt^2 + dx_1^2 + dx_2^2\right) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_7^2\,, \tag{1.1.22}$$

that is the metric of

$$AdS_4 \times S^7 \tag{1.1.23}$$

with

$$\frac{1}{2}R = \frac{\left(32\pi^2 N\right)^{1/6}}{2} l_p \tag{1.1.24}$$

curvature radius of AdS_4 and R curvature radius of S^7.

Similarly to the case of $AdS_5 \times S^5$, an M–theory excitation has $E_p \sim \frac{1}{l_p}$, and a near horizon excitation, as seen from infinity, has

$$E = f^{-1/3} E_p \sim \frac{r^2}{l_p^2} E_p \sim \frac{r^2}{l_p^3}\,. \tag{1.1.25}$$

The limit (1.1.19) is satisfied if

$$\frac{r}{l_p} \ll 1\,. \tag{1.1.26}$$

The coordinate redefinition is

$$U \equiv \frac{2r^2}{\sqrt{32\pi^2 N l_p^3}} = \frac{2r^2}{R^3}\,, \tag{1.1.27}$$

in terms of U the energies of the M–theory excitations are finite, and the metric (1.1.22) becomes

$$ds^2 = \frac{\left(32\pi^2 N\right)^{1/3} l_p^2}{4}\left[U^2\left(-dt^2 + dx_1^2 + dx_2^2\right) + \frac{dU^2}{U^2} + 4d\Omega_7^2\right]. \tag{1.1.28}$$

The near horizon region can be rescaled to the entire anti–de Sitter space.

The low energy theory, then, consists on these two decoupled systems, the free empty eleven dimensional supergravity and a system that

- from the macroscopic point of view is M–theory on $AdS_4 \times S^7$,
- from the microscopic point of view is the quantum theory on the $M2$–brane world-volume.

The Maldacena conjecture states that these two systems are equivalent.

We have taken N finite. If, instead, we take

$$N \longrightarrow \infty\,, \tag{1.1.29}$$

we have the supergravity limit. In fact,

$$R \sim N^{1/6} l_p \gg l_p\,, \tag{1.1.30}$$

and the higher energy M-theory excitations decouple from the supergravity (Kaluza Klein) excitations. Only the latter remain finite in terms of the coordinate U.

In this limit, the correspondence is between classical eleven dimensional supergravity on $AdS_4 \times S^7$ and a superconformal quantum field theory on the $M2$–brane. We will mainly consider this limit, that is the most firmly stated.

The theory on the brane has the same symmetry of the bulk theory, and this allows us to single it out. The isometry supergroup of the superalgebra is $Osp(8|4)$, that is the supergroup whose bosonic subgroup is $Sp(4,\mathbb{R}) \times SO(8) = SO(3,2) \times SO(8)$. It has $\mathcal{N} = 8$ supersymmetry. But $SO(3,2)$ is also the conformal group in three dimensions, and $Osp(8|4)$ is the superconformal supergroup of the $\mathcal{N} = 8$ SCFT in three dimensions with gauge group $U(N)$, that is the infrared limit of the $\mathcal{N} = 8$ SYM theory. This is the theory on the brane, dual to the bulk supergravity in the limit $N \longrightarrow \infty$.

Differently from the ten dimensional $AdS_5 \times S^5$ case, now the theory on the brane is conformal only at the infrared fixed point $g_{YM} = 0$; when $g_{YM} \neq 0$ we have a gauge theory not equivalent to any bulk theory; all the forms of the correspondence occur when $g_{YM} \longrightarrow 0$, and differ only in the N range.

All I said about the $M2$–brane is valid, with small differences, also for the $M5$–brane; in this case the near horizon geometry is $AdS_7 \times S^4$. Furthermore, the Maldacena conjecture is valid also for $D3$–branes, $M2$–branes and $M5$–branes corresponding to less symmetric supergravity configurations, giving less supersymmetric theories; I will examine these cases afterwards, in section 1.1.4, and the rest of this thesis concerns them.

1.1.3 The realization of the correspondence

After the formulation of the Maldacena conjecture, E. Witten [2] and, independently, S. Gubser, I. Klebanov and A. Polyakov [3] proposed a precise formulation of the AdS/CFT correspondence, telling in what sense the bulk and brane theories should be identified, and giving a method to calculating correlation functions of the quantum theory on the brane by classical supergravity (or string) calculations on the bulk. I will follow the line of Witten's reasoning.

Let us consider the correspondence between IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SCFT on $3+1$ dimensions. First of all, we can note that the conformal theory is not defined on $3+1$ Minkowski space $\mathcal{M}_4$, but on its compactified version $\widetilde{\mathcal{M}}_4$, that is $\mathcal{M}_4$ with some "points at infinity" added: without these points Minkowski is not a representation space of the conformal group $SO(4,2)$. The compactified Minkowski space $\widetilde{\mathcal{M}}_4$ coincides with the boundary of the AdS_5 space

$$\widetilde{\mathcal{M}}_4 \equiv \partial AdS_5. \tag{1.1.31}$$

On the other hand, we can consider string theory (or supergravity) on $AdS_5 \times S^5$ from the Kaluza Klein point of view, as a five dimensional supergravity theory on AdS_5 with compact internal space S^5.

We can rephrase the Maldacena conjecture as the correspondence between a supergravity theory (or string or M theory) on an AdS space times a compact space and a superconformal quantum field theory on the **boundary** of the AdS space itself. It becomes a bulk/boundary correspondence.

Then the equivalence between a theory on AdS_5 and a theory on ∂AdS_5 can be seen as a realization of the so–called *holographic principle* [36], which states that in a quantum gravity theory all physics within some volume can be described in terms of some theory on the boundary with less than one degree of freedom per Planck area.

On the other hand, the AdS space is very peculiar. It has a time–like boundary at spatial infinity; consequently, it is not possible to define the Cauchy problem, that is, to determine all the dynamics giving the field values on a Cauchy hypersurface, because the fields depend on their boundary values. On the contrary, if we give the boundary values of the fields and impose that they are regular on the bulk, there is an unique solution of the field equations. In this sense, the AdS space is intrinsically holographic.

We can now attempt to make more precise the Maldacena conjecture, relating the field theory on the boundary with supergravity (or string theory) on the bulk. The simplest recipe, that combines all the ingredients we have, is the following. Let us consider a field Φ on AdS_5. Its equation of motion $\Box\Phi = 0$, as I said, has an unique solution on the bulk with any given boundary values. Let Φ_0 be the restriction of Φ to the boundary ∂AdS_5. We will assume that in the correspondence between AdS_5 and conformal field theory on the boundary, Φ_0 couples to a conformal field $\mathcal{O}$, singlet under the gauge group, via a coupling

$$\int_{\widetilde{\mathcal{M}}_4} \Phi_0 \mathcal{O}. \tag{1.1.32}$$

In other words, we consider that the boundary values of string theory fields (in particular, supergravity fields) act as sources of gauge invariant operators in the field theory. From a D–brane perspective, we think of closed string states on the bulk as sourcing gauge singlet operators on the brane which originate as composite operators built form open

strings. Then Φ_0 is the current source of the quantum field $\mathcal{O}$ excitations, and we assume the generating functional of the correlation functions $\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\ldots\mathcal{O}(x_n)\rangle$,

$$Z(\Phi_0) \equiv \left\langle \exp\left(\int_{\widetilde{M}_4} \Phi_0 \mathcal{O}\right)\right\rangle_{CFT}, \tag{1.1.33}$$

to be

$$Z(\Phi_0) = \exp\left(-S\left(\Phi\left(\Phi_0\right)\right)\right). \tag{1.1.34}$$

Here $S(\Phi)$ is the action of classical supergravity on the bulk in the limit $g_s N \longrightarrow \infty$, $g_s \longrightarrow 0$, with classical string corrections if $g_s N$ finite, $g_s \longrightarrow 0$ and string loop corrections if g_s, N finite. However, I will consider in the following the classical supergravity case.

In this picture the interaction between two points of the boundary quantum theory is mediated by the bulk. An excitation on the boundary interacts with the bulk, propagating via the equation of motion $\Box\Phi = 0$, and in the same way it propagates from the bulk to another point on the boundary, as in Fig.1.1. To visualize better the system, and to do

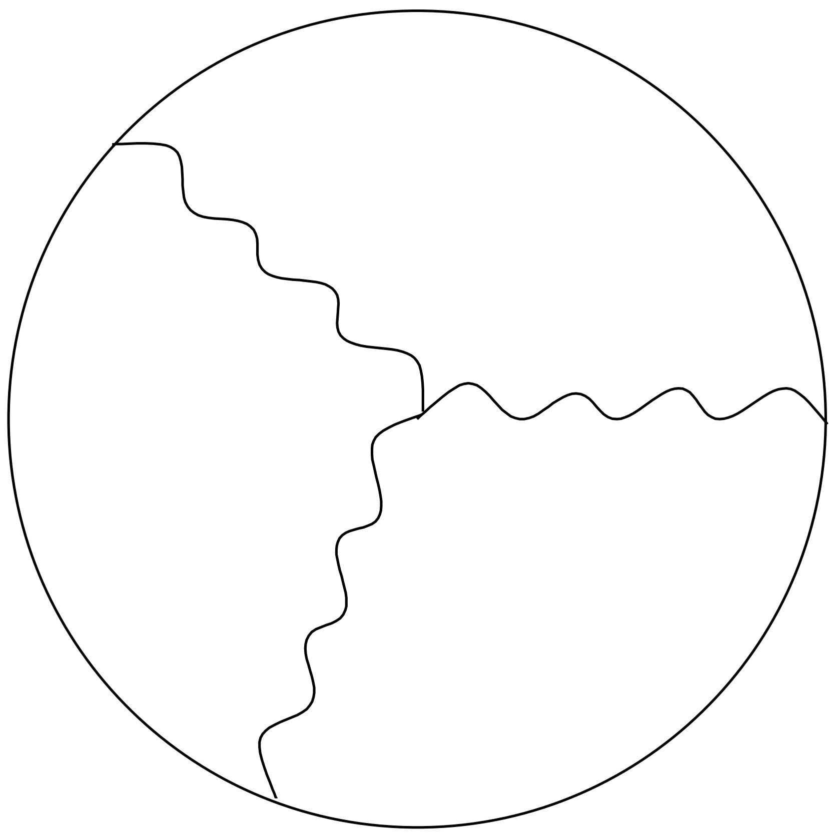


Figure 1.1: Three point function on the boundary theory via AdS/CFT correspondence

simpler calculation, it is useful to consider euclidean signature; so the AdS_5 space can be seen as the open unit ball B_5, with metric

$$ds^2 = \frac{4\sum_{a=0}^{4} dy_a^2}{\left(1-|y|^2\right)^2} \tag{1.1.35}$$

and its boundary as the sphere S^4

$$\sum_{a=0}^{4} y_a^2 = 1. \tag{1.1.36}$$

Let us consider the simplest case, the massless scalar field ϕ. Its boundary value ϕ_0 is conformally invariant, so, by conformal invariance of the action, $\mathcal{O}$ has conformal dimension $d - 1 = 4$. The equation of motion of ϕ is the Laplace equation, which can be solved with the Green function method. Doing the calculations in euclidean signature, we find [2]

$$\phi\left(x_0, x_i\right) = c \int d\mathbf{x}' \frac{x_0^4}{\left(x_0^2 + \left|\mathbf{x} - \mathbf{x}'\right|^2\right)^4} \phi_0\left(x_i'\right) \tag{1.1.37}$$

(where c is a constant depending on the normalization) and substituting this expression in the action one finds

$$I\left(\phi\right) = 2c \int d\mathbf{x} d\mathbf{x}' \frac{\phi_0\left(\mathbf{x}\right)\phi_0\left(\mathbf{x}'\right)}{\left|\mathbf{x} - \mathbf{x}'\right|^8}. \tag{1.1.38}$$

So the two point function of the operator $\mathcal{O}$ with conformal dimension 4 is proportional to $\left|\mathbf{x} - \mathbf{x}'\right|^{-8}$, and the other are zero, and this is what was expected in conformal field theory.

The same can be done for all the fields of supergravity, massless and massive. In this case the correlators are well known, and result to coincide with the ones derived with this recipe. Furthermore, one finds a relation between the masses of the fields Φ and the conformal weights of the corresponding operators $\mathcal{O}$.

To understand this, we have to define more precisely the extension of bulk fields to the boundary, first of all the metric. The metric on the open ball B_5 (1.1.35) does not extend over $\bar{B}_5$, because it becomes singular on the boundary. To get a metric which extends over $\bar{B}_5$ we have to replace (1.1.35) with a metric of the form

$$d\widetilde{s}^2 = f^2 ds^2 \tag{1.1.39}$$

with f having a zero on the boundary, for example $f = 1 - |y|^2$. Then $d\widetilde{s}^2$ restricts to a metric on the boundary S^4. As there is no natural choice of f, this metric is only well–defined up to conformal transformations: one could replace f by

$$f \longrightarrow e^w f \tag{1.1.40}$$

with w any real function on $\bar{B}_5$, and this would induce the conformal transformation

$$d\widetilde{s}^2 \longrightarrow e^{2w} d\widetilde{s}^2 \tag{1.1.41}$$

in the metric on S^5. Then the metric on AdS_5 does not define a metric on its boundary, but only a conformal structure (namely, an equivalence class of metrics). Notice that the metric on the boundary has conformal weight -2, the contravariant metric has conformal weight 2, and the corresponding operator $\mathcal{O}$ (with covariant indices) has by (1.1.32) conformal weight $4 - 2 = 2$.

[2] In these coordinates one regards the euclidean AdS_5 not as the open unit ball, but as an infinite open half-space, with boundary S^5.

Now let us consider the massive scalar fields. Differently from the massless fields, they diverge on the boundary. Their asymptotic behaviour is $\phi \sim e^{\lambda_+ z}$ where z is a coordinate that goes to infinity on the boundary, and λ_+ is the positive root of the equation [3]

$$m^2 = 16\,(\lambda + 1)\,(\lambda + 3)\,. \tag{1.1.42}$$

Then we have to take solutions of the field equation with asymptotic behaviour

$$\phi \sim f^{-\lambda_+}\phi_0 \tag{1.1.43}$$

where f is a function with a zero at the boundary, and ϕ_0 is a function on the boundary. So, like the metric, even ϕ_0 is not univocally defined, it depends on the choice of the function f (that we can assume to be the same function defining the metric on the boundary). ϕ_0, then, is a conformal field, which under conformal transformations becomes

$$\phi_0 \longrightarrow e^{w\lambda_+}\phi_0 \tag{1.1.44}$$

and has then conformal weight $-\lambda_+$. Consequently, the corresponding operator $\mathcal{O}$ of (1.1.32) has conformal weight $\Delta \equiv 4 + \lambda_+$. The relation between the mass of the bulk field ϕ and the conformal weight of the corresponding boundary operator is

$$m^2 = 16\,(\Delta - 1)\,(\Delta - 3)\,. \tag{1.1.45}$$

But representation theory of AdS_5 space tells us that the energy of an AdS_5 field is related to its mass by

$$m^2 = 16\,(E - 1)\,(E - 3)\,, \tag{1.1.46}$$

then *the energy of the bulk field coincides with the conformal weight of the corresponding boundary operator*

$$E = \Delta. \tag{1.1.47}$$

In the next chapter I will give a deeper explanation of this result. It refers to all the string theory fields Φ, and to all the theories dual by Maldacena conjecture. We can then give the more complete formulation of the AdS/CFT **correspondence**:

> *Every string theory on a background of the form*
>
> $$AdS_d \times X_{10-d} \tag{1.1.48}$$
>
> *or M–theory on a background of the form*
>
> $$AdS_d \times X_{11-d} \tag{1.1.49}$$
>
> *(were d and the compact space X_{D-d} are such that $AdS_d \times X_{D-d}$ is a supergravity solution) is equivalent to a superconformal quantum field theory on the boundary on the AdS_d space. There is a one-to-one correspondence between the on–shell fields on the bulk theory and the off-shell conformal operators (which are gauge singlets) on the boundary theory; they have the same quantum numbers, and the energy of each bulk field is equal to the conformal weight of the corresponding boundary operator. In the limit $g_s N \longrightarrow \infty$, $g_s \longrightarrow 0$ for string theory and $N \longrightarrow \infty$ for M–theory, the bulk theory reduces to classical supergravity. The generating functional of the boundary theory is given by the expression (1.1.34) in terms of the bulk theory.*

[3] with the normalization of [37], differing from the normalization of [2] and [28] by a factor 16

1.1.4 Comparison with "experiment"

The first check of the AdS/CFT correspondence has been done in [2] and [38], where it has been shown the duality, in the limit

$$g_s N \longrightarrow \infty, \quad g_s \longrightarrow 0 \tag{1.1.50}$$

for the bulk theory and

$$g_{YM} \longrightarrow 0, \quad g_{YM} N \longrightarrow \infty \tag{1.1.51}$$

for the boundary theory, between $AdS_5 \times S^5$ supergravity and $\mathcal{N} = 4$ $U(N)$ SYM theory on $\widetilde{M}_4$.

The Kaluza Klein spectrum of $AdS_5 \times S^5$ supergravity has been worked out long ago [39]. There are only the so–called "short" multiplets (see the next chapter) of five dimensional supergravity, which are protected by supersymmetry against quantum and stringy corrections [4]. The conformal fields that correspond to these excitations are similarly in "small" representations, with dimensions protected against quantum corrections.

The $\mathcal{N} = 4$ $U(N)$ SYM (that is a superconformal theory) is well known [40] in the weak coupling limit. But we need information about its strong coupling limit (1.1.51) to compare with the bulk supergravity. Fortunately, some information is protected against quantum corrections, and then does not change as the coupling varies. First of all, there are operators in "small" representations of the superconformal group. In the case of $AdS_5 \times S^5$, all the operators dual to supergravity are protected, and can then be compared. Furthermore, some correlation functions are also protected against quantum corrections and do not depend on the coupling; they can then be compared with the expression predicted by AdS/CFT correspondence (1.1.34). Both these tests have been worked out, the first in [2], the second in [38], and were successful.

Other checks has been done in several other cases of AdS/CFT correspondences (see the references in [28]): every time the spectrum at strong coupling is known, it corresponds to the Kaluza Klein supergravity spectrum, and every time some correlators at strong coupling are known, they coincide with the ones given by the (1.1.34).

1.2 G/H M–branes

1.2.1 More on the $AdS_4 \times X_7$ case

The duality between string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM theory can be generalized, as I said, to dualities between string theory on $AdS_5 \times X_5$ backgrounds, where X_5 is a compact space, and other conformal gauge theories, provided $AdS_5 \times X_5$ to be a supergravity solution [5]. In the same way, the duality between M–theory on $AdS_4 \times S^7$ and the infrared limit of the $\mathcal{N} = 8$ SYM theory on $\widetilde{\mathcal{M}}_3$ can be generalized to dualities between M–theory on

$$AdS_4 \times X_7 \tag{1.2.1}$$

[4] In fact, their masses (which in our normalization are expressed in units of R, $m = m_{\text{dimensional}} R$, are all $m_{dimensional} \sim 1/R$, then E and Δ do not depend on R. On the contrary, the stringy excitations have $m \sim (g_{YM} N)^{1/4}$, and decouple in the limit (1.1.51).

[5] One assumes that it is possible to define string theory around these backgrounds, even if it has not been done yet; however, the most part of the calculations are performed in the supergravity limit.

backgrounds, where X_7 is a compact space such that (1.2.1) is a supergravity solution, and other SCFTs.

In the literature of AdS/CFT correspondence, the most studied case is the correspondence between string theory on $AdS_5 \times X_5$ and four dimensional SCFT. The main reasons are that string theory allows to build the gauge theories on D–branes (while the theories on M–branes are to be guessed, or derived relating them to theories on D–branes), and that the strong coupling of four dimensional SYM theories is an obvious field of interest. However, even the case of correspondence between M–theory on $AdS_4 \times S^7$ is interesting, on one hand because three dimensional conformal theories are also interesting by themselves, on the other hand because it would be interesting to know something about M–theory, that today is rather mysterious; this could be the way to find the conformal theory intrinsic to M–branes. Furthermore, there are some results derived in the eighties on supergravity on $AdS_4 \times X_7$, that can be simply utilized in this new context. In the following, throughout all the thesis, I will consider only the correspondence between M–theory on (1.2.1) and three dimensional superconformal field theories.

The $AdS_4 \times S^7$ case has maximal supersymmetry: the theories of the correspondence have 32 supersymmetry charges, corresponding to $\mathcal{N} = 8$ four dimensional supergravity on the bulk and $\mathcal{N} = 8$ three dimensional SCFT on the boundary. On the contrary, the other $AdS_4 \times X_7$ cases are *less supersymmetric cases*.

Let G be the isometry of the X_7 space. Being

$$AdS_4 \equiv \frac{SO\,(3,2)}{SO\,(3,1)}\,, \tag{1.2.2}$$

the isometry of $AdS_4 \times X_7$ is

$$SO\,(3,2) \times G\,. \tag{1.2.3}$$

As I will explain in chapter 3, if X_7 admits $\mathcal{N}$ Killing spinors, namely, there are $\mathcal{N}$ solutions of the equation

$$\mathcal{D}_\alpha \eta\,(y) = c\tau_\alpha \eta\,(y) \tag{1.2.4}$$

(c is a constant depending on the normalization), then G has the form

$$G = SO\,(\mathcal{N}) \times G'\,. \tag{1.2.5}$$

Furthermore there is a supergravity solution with background $AdS_4 \times X_7$, called Freund Rubin solution (I will describe this solution afterwards), which is an $\mathcal{N}$ extended supergravity. Its isometry supergroup is

$$Osp\,(\mathcal{N}|4) \times G'. \tag{1.2.6}$$

I remind that $Osp\,(\mathcal{N}|4)$ is the isometry supergroup of AdS_4 supergravity (see [37]). It is the supergroup made up by its bosonic subgroup $SO(3,2) \times SO(\mathcal{N})$ and by the supercharges Q:

$$Osp\,(\mathcal{N}|4) = \left(\begin{array}{c|c} SO\,(3,2) & Q \\ \hline Q & SO\,(\mathcal{N}) \end{array} \right). \tag{1.2.7}$$

Notice that the $SO\,(\mathcal{N})$ part of G has become the $R-$symmetry of the supergravity: it has been absorbed by the supergroup. The remaining isometry, G', is an additional internal local symmetry of the four dimensional theory.

The bosonic part of (1.2.6) is the remnant of the AdS isometry in eleven dimensions, and is gauged by the fields resulting by the decomposition of the eleven dimensional massless graviton: the four dimensional massless graviton, and four dimensional massless vectors in the adjoint representation of G. The supersymmetries are gauged by $\mathcal{N}$ massless gravitinos. The fields of the four dimensional supergravity are organized in unitary irreducible representations (UIRs) (with spin not bigger than two) of $Osp\,(\mathcal{N}|4) \times G'$, which are the supermultiplets organized in G' representations.

On the other side, the corresponding operators on the boundary are organized in the same UIRs of the same supergroup $Osp\,(\mathcal{N}|4) \times G'$, that has also the interpretation of the superconformal group in three dimensions times G'. The energies of the four dimensional fields correspond to the conformal weights of the three dimensional operators. I remind, however, that the bulk fields are *on–shell*, the boundary operators are *off-shell*; notice that the degrees of freedom of an on–shell field on AdS_4 and the degrees of freedom of an off–shell field on $\widetilde{\mathcal{M}}_3$ coincide. On the other hand, also the R–symmetry (namely, the automorphism symmetry of the superalgebra) of four dimensional anti–de Sitter superspace and of three dimensional Poincaré superspace coincide, being $SO(\mathcal{N})$. Furthermore, it is worth noting that the Majorana spinors in three dimensions are half the Majorana spinors in four dimensions, so if we look at the supergroup $Osp\,(\mathcal{N}|4)$ as the bulk superisometry it has $\mathcal{N}$ fermionic generators, but if we look at it as the boundary superconformal group it has $2\mathcal{N}$ fermionic generators: $\mathcal{N}$ are the supersymmetry charges of three dimensional $\mathcal{N}$ extended SCFT, the other $\mathcal{N}$ are the special conformal supercharges.

A key point of AdS/CFT correspondence is that the superisometry of the two dual theories, in this case (1.2.6), is a local symmetry of the bulk theory, and a global symmetry of the boundary theory; in fact, on the bulk there is a supergravity theory, on the boundary a supersymmetric theory, whose only local symmetry is the gauge group. Then, a part of the superconformal symmetry $Osp\,(\mathcal{N}|4)$, the brane theory has a local symmetry, the gauge group that we call *colour*, and a global symmetry, the G' group that we call *flavour*, in analogy with ordinary QCD.

Let us consider the simplest case, the one with maximal supersymmetry,

$$AdS_4 \times S^7. \tag{1.2.8}$$

The seven sphere preserves $\mathcal{N} = 8$ supersymmetry, and

$$G = SO\,(8)\,, \tag{1.2.9}$$

then the flavour group G' group is not present. The symmetry group of the $d = 4$ supergravity is then

$$Osp\,(8|4)\,. \tag{1.2.10}$$

This theory is dual to a three dimensional $\mathcal{N} = 8$ SCFT, IR fixed point of an $\mathcal{N} = 8$ SYM theory, defined on the worldvolume of N $M2-$branes. The spectrum of this compactification has been determined, and the energies have been checked to be consistent with what we know on conformal weights of the primary conformal operators of the boundary theory [10]. This is a check of the AdS/CFT correspondence, but not the strongest possible check of this kind. In fact, the UIRs of $Osp\,(8|4)$ with spin not bigger than two are very constrained. As it happens for the case of $AdS_5 \times S^5$, there are only shortened representations, and the energy values of shortened representations (as I will explain in the next chapter) are univocally determined by the R–symmetry representations. And

there is no flavour group. Then the spectrum of $AdS_4 \times S^7$ supergravity can be deduced by an algebraic study of the UIRs of $Osp\,(8|4)$, it is not necessary to consider really the compactification of the supergravity; and $Osp\,(8|4)$ is also the symmetry of the boundary theory. Furthermore, the maximally symmetric supergravity is a theory more constrained than less supersymmetric cases.

Much more intriguing should be to check the AdS/CFT correspondence in lower supersymmetry cases, when the spectrum of the compactified supergravity is given not only by the $Osp\,(\mathcal{N}|4)$ algebra, but also by the geometry of the compactification. There are two kinds of known AdS_4/CFT_3 correspondences with $X_7 \neq S^7$:

- The orbifolds S^7/Γ, where Γ is a discrete subgroup of $SO(8)$ [11]. Such manifolds have the local geometry of S^7, and the corresponding supergravities are truncations of $\mathcal{N} = 8$ supergravity.
- Compact coset spaces
$$X_7 = \left(\frac{G}{H}\right)_7 \tag{1.2.11}$$
that are also Einstein spaces. They are not locally equivalent to S^7, and the corresponding supergravities are not truncations of $\mathcal{N} = 8$ supergravity.

The latter case is the more interesting, because it is not related with the S^7 case. It is the case of the so called $\frac{G}{H}$ $M-$**branes**, and is the one I have been studying.

1.2.2 Supergravity on $AdS_4 \times \left(\frac{G}{H}\right)_7$

If we put N M–branes on flat eleven dimensional Minkowski space, with N big, we get a system that, from the macroscopical point of view, and in the supergravity limit, looks like a p–brane solution of eleven dimensional supergravity (1.1.21), whose near horizon limit is $AdS_4 \times (G/H)_7$, and that asymptotically tends to flat space. It has been shown [41],[16] that for every supergravity solution of the form $AdS_4 \times (G/H)_7$, there is a brane solution of supergravity with the same symmetries of the former solution, whose near horizon limit is $AdS_4 \times (G/H)_7$, and whose asymptotic limit is a Ricci flat - but not flat - space, $\mathcal{C}(G/H)$, namely the *cone* on G/H
$$ds^2_{\mathcal{C}\left(\frac{G}{H}\right)} = dr^2 + r^2 ds^2_{\frac{G}{H}} \tag{1.2.12}$$
times the three dimensional Minkowski space. Notice that when $G/H = SO\,(8)\,/SO\,(7) = S^7$ the cone is the flat euclidean space, and $r = 0$ is a coordinate singularity, but in the other cases the singularity $r = 0$ is physical.

Then, if we put N M–branes not on $\mathcal{M}_{11}$ but on $\mathcal{M}_3 \times \mathcal{C}\,(G/H)$, we get on the branes a SCFT equivalent, by AdS/CFT correspondence, to the supergravity solution given by the near horizon geometry blown up to all the space. This supergravity solution has been found in the eighties, it is called *Freund Rubin solution* [42]:
$$\begin{array}{rclcrcl} g_{\mu\nu}\,(x,y) & = & g^0_{\mu\nu}\,(x) & \qquad & F_{\mu\nu\rho\sigma} & = & e\sqrt{g^0}\varepsilon_{\mu\nu\rho\sigma} \\ g_{\alpha\beta}\,(x,y) & = & g^0_{\alpha\beta}\,(y) & & \text{other } F & = & 0 \\ g_{\mu\alpha} & = & 0 & & \psi_\mu = \psi_\alpha & = & 0 \end{array} \tag{1.2.13}$$

where x^μ, $\mu = 0, \ldots, 3$ are the coordinates of AdS_4 space, y^α, $\alpha = 1, \ldots, 7$ are the coordinates of the internal G/H space, $g^0_{\mu\nu}$ is the AdS_4 metric, $g^0_{\alpha\beta}$ is the G/H metric.

The parameter e here introduced is related to the anti–de Sitter Radius by

$$R_{AdS} = \frac{1}{4e}. \tag{1.2.14}$$

As I said, when one performs explicit calculations, usually [28], [24], [43], [44], [45] measures dimensionful physical quantities in terms of the scale length which, in our case, is the anti–de Sitter radius; that is, setting $R_{AdS} = 1$. However, here I follow the conventions of [37], [23], where $e = 1$ and then $R_{AdS} = 1/4$. This is the reason for the discrepancy by a factor 4 in the mass normalizations of these papers. Notice that this does not mean that the parameter e is dimensionful; as I will explain in section 4.4.1, we get rid of dimensionful quantities by putting to one

$$\kappa^2 = 8\pi G_{11} \sim l_p^9; \tag{1.2.15}$$

reinstalling κ, the relation between e and anti–de Sitter radius is [6]

$$R_{AdS} = \frac{\kappa^{2/9}}{4e}. \tag{1.2.16}$$

For every seven dimensional compact coset space that is also an Einstein space, the (1.2.13) is a classical solution of eleven dimensional supergravity, and then it is possible a Kaluza Klein dimensional reduction to four dimensional supergravity. If G/H admits $\mathcal{N}$ Killing spinors, the four dimensional theory is an $\mathcal{N}$–extended supergravity (see chapter 3). The coset manifolds giving a supersymmetric Freund Rubin compactification have been completely classified in the eighties [46], [37]:

space	$\mathcal{N}$	G'
$S^7 = \frac{SO(8)}{SO(7)}$	8	
$S^7_{\text{squashed}} = \frac{SO(5)\times SO(3)\times SO(2)}{SO(3)\times SO(3)\times SO(2)}$	1	$SO(5) \times SO(3)$
$N^{0p0} = \frac{SU(3)\times SU(2)}{SU(2)\times U(1)}$	3	$SU(3)$
$M^{ppr} = \frac{SU(3)\times SU(2)\times U(1)}{SU(2)\times U(1)\times U(1)}$	2	$SU(3) \times SU(2)$
$Q^{ppp} = \frac{SU(2)\times SU(2)\times SU(2)\times U(1)}{U(1)\times U(1)\times U(1)}$	2	$SU(2) \times SU(2) \times SU(2)$
$V_{5,2} = \frac{SO(5)\times U(1)}{SO(3)\times U(1)}$	2	$SO(5)$

(1.2.17)

Most of these spaces are described in chapter 3, where the mass spectra of supergravity on $AdS_4 \times (G/H)_7$ in the cases $(G/H)_7 = M^{111}$ ($\mathcal{N} = 2$), $(G/H)_7 = N^{010}$ ($\mathcal{N} = 3$) [7] are given and, for M^{111}, explicitly worked out. The case $(G/H)_7 = V_{5,2}$ has been recently studied in [27].

[6]These formulas and conventions was derived in the context of eleven dimensional supergravity and M–theory, before the AdS/CFT correspondence was proposed. $\frac{R_{AdS}}{l_p}$ is a free parameter in the context of eleven dimensional supergravity. However, in the context of AdS/CFT correspondence, such a quantity is related to N. Then I don't give an explicit expression of κ: different conventions ($e = 1$, $e = 1/4$) correspond to different values of κ^2/l_p^9.

[7]and partially of $(G/H)_7 = Q^{111}$ ($\mathcal{N} = 2$)

Chapter 2

Representation theory of $Osp\,(\mathcal{N}|4)$

A field on four dimensional anti–de Sitter space

$$AdS_4 \equiv \frac{SO\,(3,2)}{SO\,(3,1)} \tag{2.0.1}$$

is an unitary irreducible representation (UIR) of the isometry group $SO\,(3,2)$. Notice that such representations cannot be finite dimensional, being $SO\,(3,2)$ non compact, and then have to be fields.

Supergravity on AdS_4 is defined on the $\mathcal{N}$ extended anti–de Sitter superspace, which, in the coset space formulation, is

$$AdS_{4|\mathcal{N}} \equiv \frac{Osp\,(\mathcal{N}|4)}{SO\,(3,1) \times SO\,(\mathcal{N})}. \tag{2.0.2}$$

It has 4 bosonic coordinates labelling the points on AdS_4 and $4\mathcal{N}$ fermionic coordinates transforming as Majorana spinors under $SO\,(1,3)$ and as vectors under $SO\,(\mathcal{N})$. Its isometry supergroup is $Osp\,(\mathcal{N}|4)$, so the superfields are UIRs of such a supergroup. In other words, an UIR of $Osp\,(\mathcal{N}|4)$ is a *supermultiplet* of AdS_4 fields.

The $Osp\,(\mathcal{N}|4)$ supergroup is described in the next section. Here I stress that its bosonic subalgebra is

$$SO\,(3,2) \times SO\,(\mathcal{N})\,, \tag{2.0.3}$$

namely, the anti–de Sitter isometry times the so–called *R–symmetry.* The R–symmetry is the external automorphism algebra of the supersymmetry charges. In anti–de Sitter supersymmetry it belongs to the irreducible part of the supersymmetry algebra itself, while in Poincaré supersymmetry it does not.

In general, the R–symmetry depends on the kind of supersymmetry (Poincaré or anti–de Sitter) and on the dimensionality of the theory:

	$d=3$	$d=4$	$d=5$
Poincaré	$SO\,(\mathcal{N})$	$SU\,(\mathcal{N})$	$Usp\,(\mathcal{N})$
AdS	$SO\,(\mathcal{N}_L) \times SO\,(\mathcal{N}_R)$	$SO\,(\mathcal{N})$	$SU\,(\mathcal{N}/2)$

(2.0.4)

R-symmetry

The same supergroup $Osp\,(\mathcal{N}|4)$ has also another interpretation: it is the conformal supergroup of a three dimensional Poincaré theory with $\mathcal{N}$ extended supersymmetry. In

this context, $SO(3,2)$ is the conformal group in three dimensions. The fourth coordinate translation is interpreted as conformal scaling, and the Lorentz rotations involving this coordinate are interpreted as conformal boosts. Half the fermionic generators are the supersymmetry charges, the other half are the special conformal supercharges. The R–symmetry of three dimensional Poincaré theories, like those of four dimensional AdS theories, is $SO(\mathcal{N})$. Then, the UIRs of $Osp(\mathcal{N}|4)$ can also be organized as supermultiplets of three dimensional conformal fields, namely, as three dimensional conformal superfields.

In order to make the comparison between compactified supergravity on the bulk and superconformal field theory on the boundary explicit, we need a general vocabulary between these two descriptions of $Osp(\mathcal{N}|4)$. We need to know, given a state on the bulk, which should be the corresponding conformal operator on the boundary, in order to check whether it is actually present. This is the main aim of the present chapter.

In section 1 the $osp(\mathcal{N}|4)$ superalgebra is defined with its basic properties and the conventions are established. Furthermore, its compact and non compact five–grading structures, fundamental in understanding the algebraic basis of AdS_4/CFT_3 correspondence, are described. In section 2 the theory of $SO(3,2)$ UIRs is briefly sketched. In section 3 we extend our analysis to the UIRs of $Osp(\mathcal{N}|4)$, stressing both the interpretations of $Osp(\mathcal{N}|4)$ as isometry of four dimensional anti–de Sitter superspace and as the superconformal group in three dimensions. In section 4 the method of explicit construction of $Osp(\mathcal{N}|4)$ UIRs as supergravity supermultiplets is given, and these latter are explicitly retrieved in the cases of $\mathcal{N}=1$, $\mathcal{N}=2$, and $\mathcal{N}=3$ supersymmetry. In section 5 the superspace on the bulk and on the boundary of AdS_4 is constructed, and, in the case $\mathcal{N}=2$, the short superfields are found to correspond with the $Osp(2|4)$ UIRs derived in the precedent section. Part of the content of the present chapter refers to results obtained within the collaborations [14],[15].

2.1 The $osp(\mathcal{N}|4)$ superalgebra: definition, properties and notations

The non compact superalgebra $osp(\mathcal{N}|4)$ relevant to the AdS_4/CFT_3 correspondence is a real section of the complex orthosymplectic superalgebra $osp(\mathcal{N}|4,\mathbb{C})$ that admits the Lie algebra

$$g_{even} = sp(4,\mathbb{R}) \oplus so(\mathcal{N},\mathbb{R}) \tag{2.1.1}$$

as even subalgebra. Alternatively, due to the isomorphism $sp(4,\mathbb{R}) \equiv usp(2,2)$ we can take a different real section of $osp(\mathcal{N}|4,\mathbb{C})$ such that the even subalgebra is:

$$g_{even} = usp(2,2) \oplus so(\mathcal{N},\mathbb{R})\,. \tag{2.1.2}$$

Here we rely on the second formulation (2.1.2) which is more convenient to discuss unitary irreducible representations. The two formulations are related by a unitary transformation that, in spinor language, corresponds to a different choice of the gamma matrix representation. Formulation (2.1.1) is obtained in a Majorana representation where all the gamma matrices are real (or purely imaginary), while formulation (2.1.2) is related to a Dirac representation.

Our choice for the gamma matrices in a Dirac representation is the following one[1]:

$$\Gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \qquad \Gamma^{1,2,3} = \begin{pmatrix} 0 & \tau^{1,2,3} \\ -\tau^{1,2,3} & 0 \end{pmatrix}, \qquad C_{[4]} = i\Gamma^0\Gamma^3, \tag{2.1.3}$$

having denoted by $C_{[4]}$ the charge conjugation matrix in 4–dimensions $C_{[4]}\,\Gamma^\mu\,C_{[4]}^{-1} = -(\Gamma^\mu)^T$.

Then the $Osp(\mathcal{N}|4)$ superalgebra is defined as the set of graded $(4+\mathcal{N})\times(4+\mathcal{N})$ matrices μ that satisfy the following two conditions:

$$\begin{aligned} \mu^T \begin{pmatrix} C_{[4]} & 0 \\ 0 & \mathbb{1}_{\mathcal{N}\times\mathcal{N}} \end{pmatrix} + \begin{pmatrix} C_{[4]} & 0 \\ 0 & \mathbb{1}_{\mathcal{N}\times\mathcal{N}} \end{pmatrix} \mu &= 0 \\ \mu^\dagger \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{\mathcal{N}\times\mathcal{N}} \end{pmatrix} + \begin{pmatrix} \Gamma^0 & 0 \\ 0 & -\mathbb{1}_{\mathcal{N}\times\mathcal{N}} \end{pmatrix} \mu &= 0 \end{aligned} \tag{2.1.4}$$

the first condition defining the complex orthosymplectic algebra, the second condition defining the real section with even subalgebra as in eq.(2.1.2). Eq.s (2.1.4) are solved by setting:

$$\mu = \begin{pmatrix} \varepsilon^{AB}\,\frac{1}{4}\,[\mathrm{I\!\Gamma}_A\,,\,\mathrm{I\!\Gamma}_B] & \epsilon^i \\ \bar{\epsilon}^i & \mathrm{i}\,\varepsilon_{ij}\,t^{ij} \end{pmatrix}. \tag{2.1.5}$$

In eq.(2.1.5) $\varepsilon_{ij} = -\varepsilon_{ji}$ is an arbitrary real antisymmetric $\mathcal{N}\times\mathcal{N}$ tensor, $t^{ij} = -t^{ji}$ is the antisymmetric $\mathcal{N}\times\mathcal{N}$ matrix:

$$(t^{ij})_{\ell m} = \mathrm{i}\left(\delta^i_\ell\delta^j_m - \delta^i_m\delta^j_\ell\right) \tag{2.1.6}$$

namely a standard generator of the $SO(\mathcal{N})$ Lie algebra,

$$\mathrm{I\!\Gamma}_A = \begin{cases} \mathrm{i}\,\Gamma_5\Gamma_\mu & A=\mu=0,1,2,3 \\ \Gamma_5 \equiv \mathrm{i}\Gamma^0\Gamma^1\Gamma^2\Gamma^3 & A=4 \end{cases} \tag{2.1.7}$$

denotes a realization of the $SO(2,3)$ Clifford algebra:

$$\begin{aligned} \{\mathrm{I\!\Gamma}_A\,,\,\mathrm{I\!\Gamma}_B\} &= 2\eta_{AB} \\ \eta_{AB} &= \mathrm{diag}(+,-,-,-,+) \end{aligned} \tag{2.1.8}$$

and

$$\epsilon^i = C_{[4]}\left(\bar{\epsilon}^i\right)^T \quad (i=1,\ldots\mathcal{N}) \tag{2.1.9}$$

are $\mathcal{N}$ anticommuting Majorana spinors.

The index conventions we have so far introduced can be summarized as follows. Capital indices $A,B = 0,1,\ldots,4$ denote $SO(2,3)$ vectors. The latin indices of type $i,j,k = 1,\ldots,\mathcal{N}$ are $SO(\mathcal{N})$ vector indices. The indices $a,b,c,\ldots = 1,2,3$ are used to denote spatial directions of AdS_4: $\eta_{ab} = \mathrm{diag}(-,-,-)$, while the indices of type $m,n,p,\ldots = 0,1,2$ are space-time indices for the Minkowskian boundary $\partial\,(AdS_4)$: $\eta_{mn} = \mathrm{diag}(+,-,-)$.

[1] we adopt as explicit representation of the $SO(3)$ τ matrices a permutation of the canonical Pauli matrices σ^a: $\tau^1 = \sigma^3$, $\tau^2 = \sigma^1$ and $\tau^3 = \sigma^2$.

To write the $osp(\mathcal{N}|4)$ algebra in abstract form it suffices to read the graded matrix (2.1.5) as a linear combination of generators:

$$\mu \equiv -\mathrm{i}\varepsilon^{AB}\, M_{AB} + \mathrm{i}\varepsilon_{ij}\, T^{ij} + \bar{\epsilon}_i\, Q^i \tag{2.1.10}$$

where $Q^i = C_{[4]}\left(\overline{Q}^i\right)^T$ are also Majorana spinor operators. Then the superalgebra reads as follows:

$$\begin{aligned}
[M_{AB}\,,\,M_{CD}] &= \mathrm{i}\,(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) \\
\left[T^{ij}\,,\,T^{kl}\right] &= -\mathrm{i}\,(\delta^{jk}\,T^{il} - \delta^{ik}\,T^{jl} - \delta^{jl}\,T^{ik} + \delta^{il}\,T^{jk}) \\
\left[M_{AB}\,,\,Q^i\right] &= -\mathrm{i}\frac{1}{4}\,[\Gamma_A\,,\,\Gamma_B]\,Q^i \\
\left[T^{ij}\,,\,Q^k\right] &= -\mathrm{i}\,(\delta^{jk}\,Q^i - \delta^{ik}\,Q^j) \\
\left\{Q^{\alpha i}, \overline{Q}^j_\beta\right\} &= \mathrm{i}\delta^{ij}\frac{1}{4}\,\left[\Gamma^A\,,\,\Gamma^B\right]{}^{\alpha}{}_{\beta}M_{AB} + \mathrm{i}\delta^\alpha_\beta\, T^{ij}\,.
\end{aligned} \tag{2.1.11}$$

The form (2.1.11) of the $osp(\mathcal{N}|4)$ superalgebra coincides with that given in papers [24], [23].

In the gamma matrix basis (2.1.3) the Majorana supersymmetry charges have the following form:

$$Q^i = \begin{pmatrix} a^i_\alpha \\ \varepsilon_{\alpha\beta}\bar{a}^{\beta i} \end{pmatrix}\,, \qquad \bar{a}^{\alpha i} \equiv \left(a^i_\alpha\right)^\dagger\,, \tag{2.1.12}$$

where a^i_α are two-component $SL(2,\mathbb{C})$ spinors: $\alpha, \beta, \ldots = 1, 2$. We do not use dotted and undotted indices to denote conjugate $SL(2,\mathbb{C})$ representations; we rather use different symbols a, $\bar{a}$. Raising and lowering is performed by means of the ε-symbol:

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta\,, \qquad \psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\,, \tag{2.1.13}$$

where $\varepsilon_{12} = \varepsilon^{21} = 1$, so that $\varepsilon_{\alpha\gamma}\varepsilon^{\gamma\beta} = \delta^\beta_\alpha$. Unwritten indices are contracted with the low index at the left of the high index.

2.1.1 Compact and non compact five gradings of the $osp(\mathcal{N}|4)$ superalgebra

As it is extensively explained in [47], a non-compact group G admits unitary irreducible representations of the lowest weight type if it has a G^0 with respect to whose Lie algebra g^0 there exists a *three grading* of the Lie algebra g of G. In the case of a non–compact superalgebra the lowest weight UIRs can be constructed if the three grading is generalized to a *five grading* where the even (odd) elements are integer (half-integer) graded:

$$g = g^{-1} \oplus g^{-\frac{1}{2}} \oplus g^0 \oplus g^{+\frac{1}{2}} \oplus g^{+1}\,, \tag{2.1.14}$$

$$\left[g^k, g^l\right] \subset g^{k+l} \qquad g^{k+l} = 0 \text{ for } |k+l| > 1\,. \tag{2.1.15}$$

For the supergroup $Osp(\mathcal{N}|4)$ this grading can be made in two ways, choosing as grade zero subalgebra either the maximal compact subalgebra

$$g^0 \equiv so(3) \oplus so(2) \oplus so(\mathcal{N}) \subset osp(\mathcal{N}|4) \tag{2.1.16}$$

or the non-compact subalgebra

$$\widetilde{g}^0 \equiv so(1,2) \oplus so(1,1) \oplus so(\mathcal{N}) \subset osp(\mathcal{N}|4) \tag{2.1.17}$$

which also exists, has the same complex extension and is also maximal.

The existence of the double five–grading is the algebraic core of the AdS_4/CFT_3 correspondence. Decomposing a UIR of $Osp(\mathcal{N}|4)$ into representations of g^0 exibits its interpretation as a supermultiplet of *particles states* on the bulk of AdS_4, while decomposing it into representations of $\widetilde{g}^0$ makes explicit its interpretation as a supermultiplet of *conformal primary fields* on the boundary $\partial(AdS_4)$.

In both cases the grading is determined by the generator X of the abelian factor $SO(2)$ or $SO(1,1)$:

$$[X, g^k] = k\, g^k\,. \tag{2.1.18}$$

In the compact case (see [24]) the $SO(2)$ generator X is given by M_{04}. It is interpreted as the energy generator of the four-dimensional AdS theory. It was used in [23] and [14] for the construction of the $Osp(2|4)$ representations, yielding the long multiplets of [23] and the short and ultra-short multiplets of [14]. I repeat such decompositions here.

We call H the energy generator of $SO(2)$, L_a the rotations of $SO(3)$:

$$\begin{aligned} H &= M_{04}\,, \\ L_a &= \tfrac{1}{2}\varepsilon_{abc}\, M_{bc}\,, \end{aligned} \tag{2.1.19}$$

and $M_a^{\pm}$ the boosts:

$$\begin{aligned} M_a^+ &= -M_{a4} + iM_{0a}\,, \\ M_a^- &= M_{a4} + iM_{0a}\,. \end{aligned} \tag{2.1.20}$$

The supersymmetry generators are a^i_α and $\bar{a}^{\alpha i}$. Rewriting the $osp(\mathcal{N}|4)$ superalgebra (2.1.11) in this basis we obtain:

$$\begin{aligned}
[H, M_a^+] &= M_a^+\,, \\
[H, M_a^-] &= -M_a^-\,, \\
[L_a, L_b] &= i\,\varepsilon_{abc} L_c\,, \\
[M_a^+, M_b^-] &= 2\,\delta_{ab}\, H + 2i\,\varepsilon_{abc}\, L_c\,, \\
[L_a, M_b^+] &= i\,\varepsilon_{abc}\, M_c^+\,, \\
[L_a, M_b^-] &= i\,\varepsilon_{abc}\, M_c^-\,, \\
[T^{ij}, T^{kl}] &= -i\,(\delta^{jk}\, T^{il} - \delta^{ik}\, T^{jl} - \delta^{jl}\, T^{ik} + \delta^{il}\, T^{jk})\,, \\
[T^{ij}, \bar{a}^{\alpha k}] &= -i\,(\delta^{jk}\, \bar{a}^{\alpha i} - \delta^{ik}\, \bar{a}^{\alpha j})\,, \\
[T^{ij}, a^k_\alpha] &= -i\,(\delta^{jk}\, a^i_\alpha - \delta^{ik}\, a^i_\alpha)\,, \\
[H, a^i_\alpha] &= -\tfrac{1}{2}\, a^i_\alpha\,, \\
[H, \bar{a}^{\alpha i}] &= \tfrac{1}{2}\, \bar{a}^{\alpha i}\,, \\
[M_a^+, a^i_\alpha] &= (\tau_a)_{\alpha\beta}\, \bar{a}^{\beta i}\,, \\
[M_a^-, \bar{a}^{\alpha i}] &= -(\tau_a)^{\alpha\beta}\, a^i_\beta\,, \\
[L_a, a^i_\alpha] &= \tfrac{1}{2}\, (\tau_a)_\alpha{}^\beta\, a^i_\beta\,, \\
[L_a, \bar{a}^{\alpha i}] &= -\tfrac{1}{2}\, (\tau_a)^\alpha{}_\beta\, \bar{a}^{\beta i}\,,
\end{aligned}$$

$$
\begin{aligned}
\{a^i_\alpha, a^j_\beta\} &= \delta^{ij}\,(\tau^k)_{\alpha\beta}\, M^-_k\,, \\
\{\bar{a}^{\alpha i}, \bar{a}^{\beta j}\} &= \delta^{ij}(\tau^k)^{\alpha\beta}\, M^+_k\,, \\
\{a^i_\alpha, \bar{a}^{\beta j}\} &= \delta^{ij}\,\delta_\alpha{}^\beta\, H + \delta^{ij}\,(\tau^k)_\alpha{}^\beta\, L_k + i\,\delta_\alpha{}^\beta\, T^{ij}\,.
\end{aligned}
\tag{2.1.21}
$$

The five–grading structure of the algebra (2.1.21) is shown in fig. 2.1 .

In the superconformal field theory context we are interested in the action of the $Osp(\mathcal{N}|4)$ generators on superfields living on the minkowskian boundary $\partial(AdS_4)$. To be precise the boundary is a compactification of $d = 3$ Minkowski space and admits a conformal family of metrics $g_{mn} = \phi(z)\eta_{mn}$ conformally equivalent to the the flat Minkowski metric

$$
\eta_{mn} = (+,-,-)\,, \qquad m,n,p,q = 0,1,2\,. \tag{2.1.22}
$$

Precisely because we are interested in conformal field theories the choice of representative metric inside the conformal family is immaterial and the flat one (2.1.22) is certainly the most convenient. The requested action of the superalgebra generators is obtained upon starting from the non–compact grading with respect to (2.1.17). To this effect we define the *dilatation* $SO(1,1)$ generator D and the *Lorentz* $SO(1,2)$ generators J_m as follows:

$$
D \equiv i\, M_{34}\,, \qquad J^m = \tfrac{i}{2}\,\varepsilon^{mpq} M_{pq}\,. \tag{2.1.23}
$$

In addition we define the the $d = 3$ *translation generators* P_m and *special conformal*

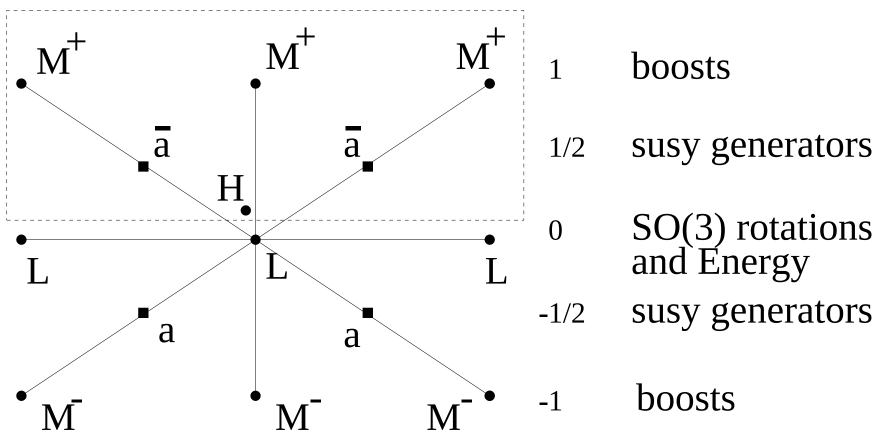


Figure 2.1: Schematic representation of the root diagram of $Osp(\mathcal{N}|4)$ in the $SO(2)\times SO(3)$ basis. The grading with respect to the energy H is given on the right.

boosts K_m as follows:

$$
\begin{aligned}
P_m &= M_{m4} - M_{3m}\,, \\
K_m &= M_{m4} + M_{3m}\,.
\end{aligned}
\tag{2.1.24}
$$

Finally we define the generators of $d = 3$ *ordinary* and *special conformal supersymmetries*, respectively given by:

$$
\begin{aligned}
q^{\alpha i} &= \tfrac{1}{\sqrt{2}}\left(a^i_\alpha + \bar{a}^{\alpha i}\right)\,, \\
s^i_\alpha &= \tfrac{1}{\sqrt{2}}\left(-a^i_\alpha + \bar{a}^{\alpha i}\right)\,.
\end{aligned}
\tag{2.1.25}
$$

The $SO(\mathcal{N})$ generators are left unmodified as above. In this new basis the $osp(\mathcal{N}|4)$-algebra (2.1.11) reads as follows

$$\begin{aligned}
[D, P_m] &= -P_m\,, \\
[D, K_m] &= K_m\,, \\
[J_m, J_n] &= \varepsilon_{mnp}\, J^p\,, \\
[K_m, P_n] &= 2\,\eta_{mn}\, D - 2\,\varepsilon_{mnp}\, J^p\,, \\
[J_m, P_n] &= \varepsilon_{mnp}\, P^p\,, \\
[J_m, K_n] &= \varepsilon_{mnp}\, K^p\,, \\
[T^{ij}, T^{kl}] &= -i\,(\delta^{jk}\, T^{il} - \delta^{ik}\, T^{jl} - \delta^{jl}\, T^{ik} + \delta^{il}\, T^{jk})\,, \\
[T^{ij}, q^{\alpha k}] &= -i\,(\delta^{jk}\, q^{\alpha i} - \delta^{ik}\, q^{\alpha j})\,, \\
[T^{ij}, s^k_\alpha] &= -i\,(\delta^{jk}\, s^i_\alpha - \delta^{ik}\, s^i_\alpha)\,, \\
[D, q^{\alpha i}] &= -\tfrac{1}{2}\, q^{\alpha i}\,, \\
[D, s^i_\alpha] &= \tfrac{1}{2}\, s^i_\alpha\,, \\
[K_m, q^{\alpha i}] &= -i\,(\gamma_m)^{\alpha\beta}\, s^i_\beta\,, \\
[P_m, s^i_\alpha] &= -i\,(\gamma_m)_{\alpha\beta}\, q^{\beta i}\,, \\
[J_m, q^{\alpha i}] &= -\tfrac{i}{2}\,(\gamma_m)^\alpha{}_\beta q^{\beta i}\,, \\
[J_m, s^i_\alpha] &= \tfrac{i}{2}\,(\gamma_m)_\alpha{}^\beta s^i_\beta\,, \\
\{q^{\alpha i}, q^{\beta j}\} &= -i\,\delta^{ij}\,(\gamma^m)^{\alpha\beta} P_m\,, \\
\{s^i_\alpha, s^j_\beta\} &= i\,\delta^{ij}\,(\gamma^m)_{\alpha\beta} K_m\,, \\
\{q^{\alpha i}, s^j_\beta\} &= \delta^{ij}\delta^\alpha{}_\beta\, D - i\,\delta^{ij}(\gamma^m)^\alpha{}_\beta J_m + i\delta^\alpha{}_\beta T^{ij}\,, \qquad (2.1.26)
\end{aligned}$$

and the five grading structure of eq.s (2.1.26) is displayed in fig.2.2. In both cases of

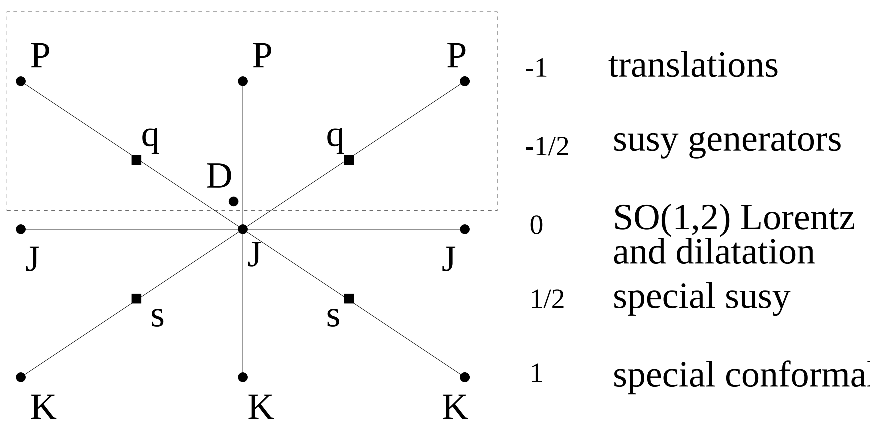


Figure 2.2: Schematic representation of the root diagram of $Osp(\mathcal{N}|4)$ in the $SO(1,1)\times SO(1,2)$ basis. The grading with respect to the dilatation D is given on the right.

fig.2.1 and fig.2.2 if one takes the subset of generators of positive grading plus the abelian grading generator $X = \left\{ \begin{array}{l} H \\ D \end{array} \right.$ one obtains a *solvable superalgebra* of dimension $4 + 2\mathcal{N}$.

2.2 UIRs of $SO(3,2)$

In order to construct the UIRs of $Osp(\mathcal{N}|4)$, the first step is to build the $SO(3,2)$ UIRs. To do it, we use the method of induced representations, using the compact grading structure; then, we are building the fields in AdS_4 space. We consider then the graded decomposition of $SO(3,2)$ with respect to its $SO(2)$ generator H,

$$so(3,2) = g_- \oplus g_0 \oplus g_+. \tag{2.2.1}$$

g_0 is the Lie algebra of the compact subgroup

$$\underbrace{SO(3)}_{\text{spin}} \times \underbrace{SO(2)}_{\text{energy}} \subset SO(3,2) \tag{2.2.2}$$

and commutes with the energy, while $g_\pm$ are raising and lowering generators with respect to H.

In practice, as we have seen, we define

$$\begin{aligned} H &= M_{04} \subset g_0 \\ L_a &= \frac{1}{2}\varepsilon_{abc} M_{bc} \subset g_0 \\ M_a^\pm &= \mathrm{i} M_{0a} \mp M_{a4} \subset g_\pm . \end{aligned} \tag{2.2.3}$$

We have

$$\begin{aligned} [H, L_a] &= 0 \\ \left[H, M_a^\pm\right] &= \pm M_a^\pm . \end{aligned} \tag{2.2.4}$$

Notice that $(H)^+ = H$, $(L_a)^+ = L_a$, $(M_a^\pm)^+ = -M_a^\mp$.

Furthermore, it is useful to organize the $M_a^\pm$ generators in the following way:

$$\begin{aligned} M^\pm_{(+)} &= \frac{1}{\sqrt{2}}\left(M_1^\pm + \mathrm{i} M_2^\pm\right) \\ M^\pm_{(-)} &= \frac{1}{\sqrt{2}}\left(M_1^\pm - \mathrm{i} M_2^\pm\right) \\ M_3^\pm &, \end{aligned} \tag{2.2.5}$$

so that they have a definite action also on the spin:

$$\begin{aligned} \left[L_3, M^\pm_{(+)}\right] &= M^\pm_{(+)} \\ \left[L_3, M^\pm_{(-)}\right] &= -M^\pm_{(-)} \\ \left[L_3, M_3^\pm\right] &= 0. \end{aligned} \tag{2.2.6}$$

We are interested into representations with energy bounded from below; then, we consider the UIRs of the compact subgroup $SO(3) \times SO(2)$ annihilated by the energy lowering generators M_a^-. We call them the ground states of the representation. Applying the raising generators M_a^+ (more precisely, the generators of the enveloping algebra of $SO(3,2)$ built

by its g_+ subspace) on the states of an $SO(3) \times SO(2)$ UIR, we get an $SO(3,2)$ UIR, and in this way one finds all the UIRs of $SO(3,2)$.

A representation of $SO(3) \times SO(2)$ is defined by the labels (E, s), where the eigenvalue of H is E and the eigenvalue of L^2 is $s(s+1)$. Its states are labeled by the L_3 eigenvalue $m = -s, \ldots, s$. Then the values of E, s (the energy and spin of the ground states) define the generic UIR of $SO(3,2)$.

We denote a generic state with the quantum numbers of $H,\ L^2,\ L_3,\ \left(\bar{E}, \bar{s}, \bar{m}\right)$, and with the quantum numbers (E, s) that single out the $SO(3,2)$ UIR to which it belongs (that is, the H, L^2 quantum numbers of the ground states of that representation):

$$\left|(E, s)\, \bar{E}, \bar{s}, \bar{m}\right\rangle . \tag{2.2.7}$$

We denote an UIR of $SO(3,2)$ with ground states having E, s by

$$D(E, s) . \tag{2.2.8}$$

2.2.1 Unitarity bounds, massless representations and singletons

A representation $D(E, s)$ is well defined only if the Hilbert space does not contain negative norm states; if it contains null norm states, the physical Hilbert space is the quotient space between the complete space and the space of the null norm states. Evaluating the norms of the excited states one finds [48] that the necessary and sufficient conditions for the absence of negative norm states are:

$$\begin{array}{ll} E \geq s+1 & \text{if } s \geq 1 \\ E \geq s+\frac{1}{2} & \text{if } s = 0, \frac{1}{2} \end{array} . \tag{2.2.9}$$

For special values of E one finds that some of the states obtained applying the raising operators M_a^+ on the ground states have vanishing norms. This means that they are decoupled from the representation, form another UIR of $SO(3,2)$, and our UIR is **shortened**. It happens when:

$$\begin{array}{ll} E = s+1 & s \geq 1 \\ E = s+\frac{1}{2} & s = 0, \frac{1}{2} \\ & \\ \multicolumn{2}{c}{\text{Short } AdS_4 \text{ UIRs}} \end{array} . \tag{2.2.10}$$

For these values of E, s the equation of motion acquires gauge invariance; the states decoupled because of the shortening are the gauge degrees of freedom, which can be removed by gauge fixing.

The shortened representations partially coincide with the *massless representations*. It is not obvious how to define the mass in anti–de Sitter theories, since the quadratic Casimir operator $\mathcal{C} = M_{AB} M^{AB}$ is different from the usual mass $P_a P^a$, which is not a conserved quantity. The usual way to define a mass for AdS_4 UIRs [49], [48] refers to the concept of masslessness, inherited by analogy from Poincaré theories. In Poincaré space, the massless field equations have enhanced symmetry, from $ISO(3,1)$ to conformal symmetry $SO(4,2)$. Furthermore, the massless Poincaré representations are UIRs of the conformal group, irreducible under $ISO(3,1) \subset SO(4,2)$. This phenomenom occurs

also in AdS_4 space, and when it occurs we name the corresponding AdS representation massless. Another reason for this choice is that these AdS_4 representations become, by Inonü Wigner contraction, the massless Poincaré representations; indeed, the corresponding mass generator goes to zero in this limit. With this definition, the massless AdS_4 representations are the following:

$$\begin{array}{ll} E = s+1 & s \geq \frac{1}{2} \\ E = 1, 2 & s = 0 \end{array} \qquad \text{Massless } AdS_4 \text{ UIRs} . \tag{2.2.11}$$

We can see that the $D(s+1, s)$ with $s \geq 1$ are both shortened and massless representations. For $s = 1/2, 0$, the massless representations are not shortened: they do not have gauge invariance, but their equations of motion are conformal and their contractions are Poincaré massless representations. There is only a little subtlety: the $D(s+1, s)$ $s \geq 1/2$ are UIRs of $SO(4,2)$, but $D(1,0)$ and $D(2,0)$ are not separately $SO(4,2)$ UIRs: only their direct sum $D(1,0) \oplus D(2,0)$ is an $SO(4,2)$ UIR. The Inonü Wigner contractions of $D(s+1, s)$ $s \geq 1/2$ and $D(1,0) \oplus D(2,0)$ are the Poincaré massless representations.

The shortened representations with $s = 0, 1/2$ are not massless representations. They have very peculiar properties: they do not describe a sufficient number of degrees of freedom to admit a field realization on AdS_4; once the gauge degrees of freedom are removed, the only remaining degrees of freedom live on the boundary ∂AdS_4, and not on the bulk of AdS_4 itself. These representations, found by Dirac [50], are called **singletons**:

$$\begin{array}{ll} E = 0 & s = \frac{1}{2} \\ E = \frac{1}{2} & s = 1 \end{array} \qquad \text{Singleton } AdS_4 \text{ UIRs} . \tag{2.2.12}$$

They cannot live on the bulk of AdS_4, but only on the boundary. Furthermore, the tensor products of the singletons yield all the massless AdS_4 representations.

Now that we have defined when a representation is massless, we define the squared mass as the additional constant term in the quadratic field equations,

$$\boxtimes_s^{\text{massless}} \Phi = m_{(s)}^2 \Phi . \tag{2.2.13}$$

The mass squared is linear in the quadratic Casimir $m_{(s)}^2 = \beta_{(s)} \left(\mathcal{C}_2 + \alpha_{(s)}\right)$; the overall

normalization $\beta_{(s)}$ is arbitrary. We take the normalization of [37],[14],[16], that gives [2]

$$\begin{array}{|c|rl|}
\hline
s=0 & m^2_{(0)} & = 16\left(E_{(0)}-2\right)\left(E_{(0)}-1\right) \\
\hline
s=1/2 & \left|m_{(1/2)}\right| & = 4E_{(1/2)}-6 \\
\hline
s=1 & m^2_{(1)} & = 16\left(E_{(1)}-2\right)\left(E_{(1)}-1\right) \\
\hline
s=3/2 & \left|m_{(3/2)}+4\right| & = 4E_{(3/2)}-6 \\
\hline
\end{array} \tag{2.2.14}$$

Notice that when $s=0$, for each energy value $\frac{1}{2}<E<\frac{5}{2}$ there are two mass square values; they correspond to the same form of the field equation. Furthermore, when $1<E<2$ the mass square is negative, $-4<m^2<0$; however, it has been shown [51] that in anti–de Sitter space, due to the presence of the boundary, the stability bound is, in our normalizations, $m^2>-4$ and not $m^2>0$.

2.3 UIRs of $Osp\,(\mathcal{N}|4)$ viewed in the compact and non compact five–grading bases

We start by briefly recalling the procedure of [24], [52] to construct UIRs of $Osp(\mathcal{N}|4)$ in the compact grading (2.1.16) (these procedure will be discussed more extensively in the next section). Then, in a parallel way to what was done in [47] for the case of the $SU(2,2|4)$ superalgebra we show that also for $Osp(\mathcal{N}|4)$ in each UIR carrier space there exists an unitary rotation that maps eigenstates of H, L^2, L_3 into eigenstates of D, J^2, J_2. By means of such a rotation the decomposition of the UIR into $SO(2)\times SO(3)$ representations is mapped into an analogous decomposition into $SO(1,1)\times SO(1,2)$ representations. While $SO(2)\times SO(3)$ representations describe the *on–shell* degrees of freedom of a *bulk particle* with an energy E_0 and a spin s_0, irreducible representations of $SO(1,1)\times SO(1,2)$ describe the *off-shell* degrees of freedom of a *boundary field* with scaling weight D and Lorentz character J. Relying on this we show how to construct the on-shell four-dimensional superfield multiplets that generate the states of these representations and the off-shell three-dimensional superfield multiplets that build the conformal field theory on the boundary.

Lowest weight representations of $Osp(\mathcal{N}|4)$ are constructed starting from the basis (2.1.21) and choosing a *Clifford vacuum state* such that

$$\begin{aligned}
M_i^-|(E_0,s_0,\Lambda_0)\rangle &= 0\,, \\
a^i_\alpha|(E_0,s_0,\Lambda_0)\rangle &= 0\,,
\end{aligned} \tag{2.3.1}$$

[2] In [28], [24], [43], [45] the mass normalization differs by a factor 4, for the reason explained in chapter 1.

where E_0 denotes the eigenvalue of the energy operator M_{04} while s_0 and Λ_0 are the labels of an irreducible $SO(3)$ and $SO(\mathcal{N})$ representation, respectively[3]. In particular we have:

$$
\begin{aligned}
M_{04}\,|(E_0,s_0,\Lambda_0)\rangle &= E_0\,|(E_0,s_0,\Lambda_0)\rangle \\
L_a\,L_a\,|(E_0,s_0,\Lambda_0)\rangle &= s_0(s_0+1)\,|(E_0,s_0,\Lambda_0)\rangle \\
L_3|(E_0,s_0,\Lambda_0)\rangle &= s_0\,|(E_0,s_0,\Lambda_0)\rangle\,.
\end{aligned}
\tag{2.3.2}
$$

The states filling up the UIR are then built by applying the operators M^- and the anti-symmetrized products of the operators $\bar{a}^i_\alpha$:

$$
\left(M_1^+\right)^{n_1}\left(M_2^+\right)^{n_2}\left(M_3^+\right)^{n_3}[\bar{a}^{i_1}_{\alpha_1}\dots\bar{a}^{i_p}_{\alpha_p}]|(E_0,s_0,\Lambda_0)\rangle\,. \tag{2.3.3}
$$

The antisymmetrization of the fermionic operators is due to the fact that

$$
\{\bar{a}^{\alpha i},\bar{a}^{\beta j}\}=\delta^{ij}(\tau^k)^{\alpha\beta}\,M_k^+ \tag{2.3.4}
$$

so the symmetrized fermionic generators yield excited states of the same AdS_4 fields, not new AdS_4 fields.

Lowest weight representations are similarly constructed with respect to five–grading (2.1.26). One starts from a vacuum state that is annihilated by the conformal boosts and by the special conformal supersymmetries

$$
\begin{aligned}
K_m\,|(D_0,j_0,\Lambda_0)\rangle &= 0\,,\\
s^i_\alpha\,|(D_0,j_0,\Lambda_0)\rangle &= 0\,,
\end{aligned}
\tag{2.3.5}
$$

and that is an eigenstate of the dilatation operator D and an irreducible $SO(1,2)$ representation of spin j_0:

$$
\begin{aligned}
D\,|(D_0,j_0,\Lambda_0)\rangle &= D_0\,|(D_0,j_0,\Lambda_0)\rangle\\
J_m\,J_n\,\eta^{mn}\,|(D_0,j_0,\Lambda_0)\rangle &= j_0(j_0+1)\,|(D_0,j_0,\Lambda_0)\rangle\\
J_2\,|(D_0,j_0,\Lambda_0)\rangle &= j_0|(D_0,j_0,\Lambda_0)\rangle\,.
\end{aligned}
\tag{2.3.6}
$$

As for the $SO(\mathcal{N})$ representation the new vacuum is the same as before. The states filling the UIR are now constructed by applying to the vacuum the operators P_m and the anti-symmetrized products of $q^{\alpha i}$,

$$
(P_0)^{p_0}\,(P_1)^{p_1}\,(P_2)^{p_2}\,[q^{\alpha_1 i_1}\dots q^{\alpha_q i_q}]|(D_0,j_0,\Lambda_0)\rangle\,. \tag{2.3.7}
$$

In the language of conformal field theories the vacuum state satisfying eq.(2.3.5) is named a *primary state* (corresponding to the value at $z^m=0$ of a primary conformal field). The states (2.3.7) are called the *descendants*.

The rotation between the $SO(3)\times SO(2)$ basis and the $SO(1,2)\times SO(1,1)$ basis is performed by the operator:

$$
U\equiv\exp\left[\tfrac{i}{\sqrt{2}}\pi(H-D)\right]\,, \tag{2.3.8}
$$

[3]In this context we call it state even if it is a collection of states.

which has the following properties

$$\begin{aligned} DU &= -UH\,, \\ J_0U &= i\,UL_3\,, \\ J_1U &= UL_1\,, \\ J_2U &= UL_2\,, \end{aligned} \tag{2.3.9}$$

with respect to the grade 0 generators. Furthermore, with respect to the non vanishing grade generators we have:

$$\begin{aligned} K_0U &= -i\,UM_3^-\,, \\ K_1U &= -UM_1^-\,, \\ K_2U &= -UM_2^-\,, \\ P_0U &= i\,UM_3^+\,, \\ P_1U &= UM_1^+\,, \\ P_2U &= UM_2^+\,, \\ q^{\alpha i}U &= -i\,U\bar{a}^{\alpha i} \\ s^i_\alpha U &= i\,Ua^i_\alpha\,. \end{aligned} \tag{2.3.10}$$

As one immediately sees from (2.3.10), U interchanges the compact five–grading structure of the superalgebra with its non compact one. In particular the $SO(3)\times SO(2)$-vacuum with energy E_0 is mapped into an $SO(1,2)\times SO(1,1)$ primary state and one obtains all the descendants (2.3.7) by acting with U on the particle states (2.3.3). Furthermore from (2.3.9) we read the conformal weight and the Lorentz group representation of the primary state $U|(E_0,s_0,\Lambda_0)\rangle$. Indeed its eigenvalue with respect to the dilatation generator D is:

$$D_0 = -E_0\,, \tag{2.3.11}$$

and we find the following relation between the Casimir operators of $SO(1,2)$ and $SO(3)$,

$$J^2U = UL^2\,, \qquad J^2 \equiv -J_0^2 + J_1^2 + J_2^2\,, \tag{2.3.12}$$

which implies that

$$j_0 = s_0\,. \tag{2.3.13}$$

Hence under the action of U a particle state of energy E_0 and spin s_0 of the bulk is mapped into a *primary conformal field* of conformal weight $-E_0$ and Lorentz spin s_0 on the boundary. This discussion is visualized in fig.2.3.

As in $SO\,(3,2)$ representation theory, even the $Osp\,(\mathcal{N}|4)$ UIRs have to satisfy unitarity bounds, because all the states (2.3.3) or (2.3.7) must have nonnegative norms. When some of these bounds are saturated, some norms vanish, and the corresponding representations are shortened. These representations are BPS states, namely they are protected against quantum corrections; indeed the number of the states does not change with renormalization, so the shortening condition, which is a condition on the quantum numbers E_0, s_0, Λ_0 or D_0, j_0, Λ_0, must remain satisfied; then, being Λ_0 a non renormalized quantity, also E_0 and D_0 are not renormalized.

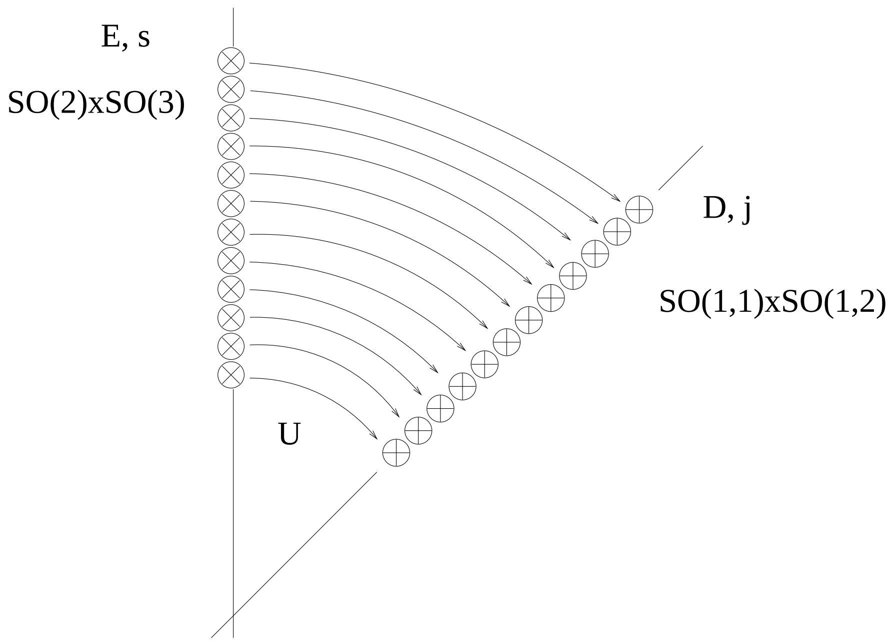


Figure 2.3: The operator $U = \exp\{\left(i\pi/\sqrt{2}\right)(H-D)\}$ rotates the Hilbert space of the physical states. It takes states labeled by the Casimirs (E, s) of the $SO(2)\times SO(3) \subset Osp(\mathcal{N}|4)$ into states labeled by the Casimirs (D, j) of $SO(1,1)\times SO(1,2)$.

2.4 The explicit construction of $Osp\,(\mathcal{N}|4)$ UIRs

Now I show how the $Osp\,(\mathcal{N}|4)$ UIRs are explicitly worked out, from the compact grading viewpoint, namely, as supermultiplets of $\mathcal{N}$ extended AdS_4 fields. I remind that we are interested only on supermultiplets not containing fields with spin greater than two.

2.4.1 Finding all the states

As I said, to find the field structure of a supermultiplet, one does not have to consider excited states of the fields: it suffices to restrict one's attention to their ground states, which are $SO\,(3)\times SO\,(2)\times SO\,(\mathcal{N})$ UIRs annihilated by the M_i^- operators. Their spin and R–symmetry labels E, s, Λ identify the corresponding AdS_4 field, namely an $SO\,(3,2)$ UIR. A supermultiplet is identified by its lowest energy AdS_4 field, whose ground states are an $SO\,(3)\times SO\,(2)\times SO\,(\mathcal{N})$ UIR $|\,(E_0, s_0, \Lambda_0)\rangle$ satisfying

$$\begin{aligned} M_i^-|(E_0, s_0, \Lambda_0)\rangle &= 0\,, \\ a_\alpha^i|(E_0, s_0, \Lambda_0)\rangle &= 0\,. \end{aligned} \tag{2.4.1}$$

We name this state the vacuum, namely, the ground state of the lowest energy $SO\,(3,2)$ representation $D\,(E_0, s_0)$ contained in the supermultiplet. The other fields $D\,(E, s)$ of the supermultiplet are found by applying the antisymmetrized products of fermionic raising generators on the vacuum. We denote the entire supermultiplet, namely, the $Osp\,(\mathcal{N}|4)$ UIR, as

$$SD\,(E_0, s_0, \Lambda_0|\mathcal{N})\,. \tag{2.4.2}$$

The first step is then to find, given $\mathcal{N}$, all the operators of the form

$$ {}^{p}K^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p} = [\bar{a}^{i_1}_{\alpha_1} \dots \bar{a}^{i_p}_{\alpha_p}]. \tag{2.4.3} $$

Then, for each vacuum state, applying on it the operators (2.4.3) we find all the possible fields of the corresponding supermultiplet. Each operator ${}^{p}K$ has given spin and R–symmetry labels, which have to be composed with that of the vacuum.

Since the $\bar{a}$ generators in the (2.4.3) are antisymmetrized, if two $\bar{a}$'s are symmetric in the $SO\left(\mathcal{N}\right)$ R–symmetry indices, they have to be antisymmetric in the $SU\left(2\right)$ spin indices. Notice that we consider $SU\left(2\right)$ instead of its locally isomorphic group $SO\left(3\right)$ as spin group, in order to make the calculation simpler: all the spin representations can be expressed by means of $SU\left(2\right)$ Young tableaux. A useful trick to find, given $\mathcal{N}$, all the possible ${}^{p}K^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p}$, is to consider temporarily the $\bar{a}^{i}_{\alpha}$ as a representation of

$$ SU\left(2\right) \times SU\left(\mathcal{N}\right) \supset SU\left(2\right) \times SO\left(\mathcal{N}\right). \tag{2.4.4} $$

Actually, they are a representation of $SU\left(2\right) \times SU\left(\mathcal{N}\right)$ only as a vector space, not as an algebra, because the superalgebra (2.1.21) is not $SU\left(\mathcal{N}\right)$–invariant. However, we can find the $\widetilde{{}^{p}K}$ operators irreducible under $SU\left(2\right) \times SU\left(\mathcal{N}\right)$ and then branch them in $SU\left(2\right) \times SO\left(\mathcal{N}\right)$ representations obtaining in this way the ${}^{p}K$ operators. The reason for this procedure is that, given an operator $\widetilde{{}^{p}K}^{i_1 \dots i_p}_{\alpha_1 \dots \alpha_p}$, if the representation of its $SU\left(\mathcal{N}\right)$ indices is described by a given Young tableau, the representation if its $SU\left(2\right)$ indices has to be described by the transposed Young tableau, in order to have a representation antisymmetric in the exchange $i_a \leftrightarrow i_b, \ \alpha_a \leftrightarrow \alpha_b$. Then if we write all the allowed $SU\left(2\right)$ Young diagrams whose transposed are allowed $SU\left(\mathcal{N}\right)$ Young diagrams, we find all the allowed operators $\widetilde{{}^{p}K}$, and decomposing them in $SU\left(2\right) \times SO\left(\mathcal{N}\right)$ irreducible representations we find all the ${}^{p}K$'s.

The states created by operators ${}^{p}K$ (containing p generators $\bar{a}$) are denoted as the B_p sector of the representation. The maximum possible value of p is $p = 2\mathcal{N}$, corresponding to the $SU(2)$ Young tableau with two rows and $\mathcal{N}$ columns.

2.4.2 Finding unitarity bounds and norms

The determination of all the possible states is the simplest part of our work. The most cumbersome calculation is the derivation of the unitarity bounds and the shortenings of the representations. This should be done by calculating the norms of the states we have found, by means of the algebra (2.1.21) which allows us to express the norms in terms of E_0, s_0, Λ_0. But this calculation is affordable only up to the B_2 sector, while when there are three or more $\bar{a}$ factors the calculation is too long. This because an operator $\bar{a}$ on a ground state of an AdS_4 field in general gives not only the ground state of another field of a multiplet, but also excited states of less energetic fields of the multiplet. We have then to use other information in order to find the structure of short multiplets:

- The $Osp\left(\mathcal{N}|4\right)$ UIRs are also UIRs of $Osp\left(\mathcal{N}'|4\right) \subset Osp\left(\mathcal{N}|4\right)$ with $\mathcal{N}' < \mathcal{N}$, so the $\mathcal{N}$–supermultiplets must be decomposable in $\mathcal{N}'$ supermultiplets, and the vanishing of some states implies the vanishing of other states.

- By Inonü Wigner contraction, the $Osp\,(\mathcal{N}|4)$ UIRs become massless representations (eventually reducible) of the Poincaré superalgebra.Then, all the $\mathcal{N}$ extended anti–de Sitter supermultiplets must be decomposable in $\mathcal{N}$ extended massless Poincaré supermultiplets; the only exception is the supersingleton representation.
- The massless UIRs of $Osp\,(\mathcal{N}|4)$ are well known, being the fields of exact four dimensional supergravity; they coincide with Poincaré massless supermultiplets. This is understandable: the exact supergravity is a field theory whose field content cannot depend on a particular vacuum, Poincaré space or anti–de Sitter space; only Kaluza Klein supergravity, which is a linearized theory, obtained as a truncation of exact massless eleven dimensional supergravity expanded around a given background, is reminiscent of that background.
- Explicit calculations of Kaluza Klein spectrum of specific $\mathcal{N}$ extended supergravities, as the ones in the next chapter, allow to fill the gaps in the knowledge on the supermultiplet structure.
- As I will show afterwards, the superfield formalism allows another formulation of short representations, which can be useful to derive them. This has not been done for all values of $\mathcal{N}$, but in the known cases is a good check of the multiplet structure.

2.4.3 The structure of $\mathcal{N} = 1$ supermultiplets

This case has been completely worked out in [52]. There is no R–symmetry group. The maximum number of fermionic generators allowed is $p = 2\mathcal{N} = 2$. There are:

- the B_0 sector, created by the identity $\mathbb{1}$, that is the lowest lying representation $D\,(E_0, s_0)$;
- the B_1 sector, created by
$$^1K_\alpha = \bar{a}_\alpha, \tag{2.4.5}$$
having spin 1/2.
- the B_2 sector, created by
$$^2K = \varepsilon^{\alpha\beta}\bar{a}_\alpha\bar{a}_\beta, \tag{2.4.6}$$
having spin 0.

There are two cases:

1. $s_0 = 0$
The lowest lying field is $D\,(E_0, 0)$, the spin 1/2 operator of B_1 gives $D\,(E_0{+}1/2,1/2)$. The spin 0 operator of B_2 gives $D\,(E_0 + 1, 0)$. The entire supermultiplet is then
$$D\,(E_0, 0) \oplus D\,(E_0 + 1/2, 1/2) \oplus D\,(E_0 + 1, 0)\,. \tag{2.4.7}$$
2. $\frac{1}{2} \geq s_0 \geq \frac{3}{2}$
The spin 1/2 operator of B_1 gives $D\,(E_0 + 1/2, s_0 + 1/2) \oplus D\,(E_0 + 1/2, s_0 - 1/2)$. The spin 0 operator of B_2 gives $D\,(E_0 + 1, s_0)$. The entire supermultiplet is then
$$D\,(E_0, s_0) \oplus D\,(E_0 + 1/2, s_0 + 1/2) \oplus D\,(E_0 + 1/2, s_0 - 1/2) \oplus D\,(E_0 + 1, s_0)\,. \tag{2.4.8}$$

We have now to carry on the calculation of the norms, using the (2.1.21) algebra. We denote the vacuum, with norm 1, by

$$|\Omega\rangle \equiv |\,(E_0, s_0)\;\; E_0, s_0, m\rangle. \tag{2.4.9}$$

$\boxed{s = 0 \text{ case}}$

- B_1 sector
  The operator $\bar{a}_1$ gives

$$\bar{a}_1|\Omega\rangle = R_{1/2}|\,(E_0 + 1/2, 1/2)\;\; E_0 + 1/2, 1/2, 1/2\rangle\,. \tag{2.4.10}$$

  Using the algebra (2.1.21), we find

$$\langle\Omega|a_1\bar{a}_1|\Omega\rangle = E_0 = \left|R_{1/2}\right|^2, \tag{2.4.11}$$

  that yields the condition $E_0 \geq 0$. The condition arising from $\bar{a}_2$ is identical.

- B_2 sector
  The operator $\varepsilon^{\alpha\beta}\bar{a}_\alpha\bar{a}_\beta$ gives

$$\begin{aligned} \left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= R_0|\,(E_0+1,0)\,E_0+1,0,m\rangle + \\ &\quad +\beta M_3^+|\,(E_0,0)\,E_0,0,0\rangle \end{aligned} \tag{2.4.12}$$

  Let us determine β.

$$\begin{aligned} M_3^-\left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= \left[M_3^-\left[\bar{a}_1, \bar{a}_2\right]\right]|\Omega\rangle = 0 = \\ &= -2E_0\beta|\,(E_0,0)\,E_0,0,0\rangle, \end{aligned} \tag{2.4.13}$$

  then $\beta = 0$. Now we can find $|R_0|^2$:

$$\langle\Omega|\left[a_2, a_1\right]\left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle = 4E_0^2 - 2E_0 = |R_0|^2\,. \tag{2.4.14}$$

  The unitarity condition arising from this sector is then

$$E_0 \geq \frac{1}{2}, \tag{2.4.15}$$

  stronger than the condition arising from the B_1 sector. This is then the only unitarity condition of the representation. When it is saturated, $R_0 = 0$ and $D\,(E_0+1,0)$ decouples. Furthermore, in this case ($E_0 = 1/2$) the representations $D\,(E_0,0)$ and $D\,(E_0+1/2,1/2)$ are the singletons.

$\boxed{s > 0 \text{ case}}$

- B_1 sector
  The operator $\bar{a}_1$ gives

$$\begin{aligned} \bar{a}_1|\Omega\rangle &= R_{1/2}\sqrt{\frac{s_0+m+1}{2s_0+1}}|\,(E_0+1/2, s_0+1/2)\;\; E_0+1/2, s_0+1/2, m+1/2\rangle + \\ &\quad -R_{-1/2}\sqrt{\frac{s_0-m}{2s_0+1}}|\,(E_0+1/2, s_0-1/2)\;\; E_0+1/2, s_0-1/2, m-1/2\rangle. \end{aligned} \tag{2.4.16}$$

To find the values of the constants $R_{1/2}, R_{-1/2}$, we take specific values of m and determine the norms using the algebra (2.1.21).
Taking $m = s$,

$$\langle\Omega|a_1\bar{a}_1|\Omega\rangle = E_0 + s_0 = \left|R_{1/2}\right|^2, \tag{2.4.17}$$

that yields the condition $E_0 + s_0 \geq 0$.
Taking $m = -1/2$,

$$\langle\Omega|a_1\bar{a}_1|\Omega\rangle = E_0 - 1/2 = \frac{1}{2}\left(\left|R_{1/2}\right|^2 + \left|R_{-1/2}\right|^2\right), \tag{2.4.18}$$

that yields the condition $E_0 - s_0 - 1 \geq 0$.
The unitarity condition arising from B_1 is then

$$E_0 \geq s_0 + 1. \tag{2.4.19}$$

The condition arising from $\bar{a}_2$ is identical. This condition coincides with the unitarity bound of AdS_4 fields for $s_0 > 1$, and is stronger for $s_0 = 1/2$. When it is saturated, $R_{-1/2} = 0$ and the $D\left(E_0 + 1/2, s_0 - 1/2\right)$ field decouples. Furthermore, $E_0 = s_0 + 1$ is the masslessness condition for AdS_4 fields with $s_0 \geq 1/2$, so when it is saturated the fields $D\left(E_0, s_0\right)$ and $D\left(E_0 + 1/2, s_0 + 1/2\right)$ in (2.4.28) are massless.

- B_2 sector
  The operator $\varepsilon^{\alpha\beta}\bar{a}_\alpha\bar{a}_\beta$ gives

$$\begin{aligned} \left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= R_0|\left(E_0 + 1, s_0\right)E_0 + 1, s_0, m\rangle + \\ &\quad + \text{boosted elements of } D\left(E_0, s_0\right). \end{aligned} \tag{2.4.20}$$

In order to calculate $|R_0|^2$ we take $m = s_0$. Then

$$\begin{aligned} \left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= R_0|\left(E_0 + 1, s_0\right)E_0 + 1, s_0, s_0\rangle + \\ &\quad + \alpha M^+_{(+)}|\left(E_0, s_0\right)E_0, s_0, s_0 - 1\rangle + \\ &\quad + \beta M^+_3|\left(E_0, s_0\right)E_0, s_0, s_0\rangle. \end{aligned} \tag{2.4.21}$$

Let us determine α and β.

$$\begin{aligned} M^-_{(-)}\left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= \left[M^-_{(-)}\left[\bar{a}_1, \bar{a}_2\right]\right]|\Omega\rangle = \\ &= \sqrt{2}J_-|\Omega\rangle = 2\sqrt{s_0}|\left(E_0, s_0\right)E_0, s_0, s_0 - 1\rangle = \\ &= \left(-2\left(E_0 + s_0 - 1\right)\alpha + 2\sqrt{s_0}\beta\right)|\left(E_0, s_0\right)E_0, s_0, s_0 - 1\rangle \end{aligned} \tag{2.4.22}$$

$$\begin{aligned} M^-_3\left[\bar{a}_1, \bar{a}_2\right]|\Omega\rangle &= \left[M^-_3\left[\bar{a}_1, \bar{a}_2\right]\right]|\Omega\rangle = \\ &= 2s_0|\left(E_0, s_0\right)E_0, s_0, s_0\rangle = \\ &= \left(2\sqrt{s_0}\alpha - 2E_0\beta\right)|\left(E_0, s_0\right)E_0, s_0, s_0\rangle, \end{aligned} \tag{2.4.23}$$

then

$$\begin{aligned} 2\alpha\sqrt{s_0} - 2E_0\beta &= 2s_0 \\ -\alpha\left(E_0 + s_0 - 1\right) + \beta\sqrt{s_0} &= \sqrt{s_0} \end{aligned} \tag{2.4.24}$$

which gives

$$\begin{aligned}
\alpha &= -\frac{\sqrt{s_0}}{E_0 - 1} \\
\beta &= -\frac{s_0}{E_0 - 1}.
\end{aligned} \tag{2.4.25}$$

Now we can find $|R_0|^2$:

$$\begin{aligned}
&\langle\Omega|\,[a_2, a_1]\,[\bar{a}_1, \bar{a}_2]\,|\Omega\rangle = \\
&= 4E_0^2 - 2E_0 - 4s_0\,(s_0 + 1) = |R_0|^2 + 2\alpha^2\,(E_0 + s_0 - 1) + 2\beta^2 E_0 - 4\alpha\beta\sqrt{s_0} = \\
&= |R_0|^2 + 2s\tfrac{E_0 + s_0 - 1}{(E_0-1)^2} + \tfrac{2s_0^2 E_0}{(E_0-1)^2} - \tfrac{4s^2}{(E_0-1)^2},
\end{aligned} \tag{2.4.26}$$

that with some elementary but tedious manipulation gives

$$|R_0|^2 = \frac{2}{E_0 - 1}\,(2E_0 - 1)\,(E_0 + s_0)\,(E_0 - s_0 - 1)\,. \tag{2.4.27}$$

The condition (2.4.19) guarantees $|R_0|^2 \geq 0$, and then is the only unitarity condition of the representation. When it is saturated, $R_0 = 0$ and $D\,(E_0 + 1, s_0)$ decouples.

In conclusion, the complete list of $Osp\,(1|4)$ UIRs is:

1. $E_0 > s_0 + 1,\ \frac{1}{2} < s_0 < \frac{3}{2}$: massive vector ($s_0 = 1/2$), gravitino ($s_0 = 1$) and graviton ($s_0 = 3/2$) multiplets

$$\begin{aligned}
SD\,(E_0, s_0|1) &= D\,(E_0, s_0) \oplus D\,(E_0 + 1/2, s_0 + 1/2) \oplus \\
&\quad \oplus D\,(E_0 + 1/2, s_0 - 1/2) \oplus D\,(E_0 + 1, s_0)\,.
\end{aligned} \tag{2.4.28}$$

   The fields of these multiplets are all massive.

2. $E_0 = s_0 + 1,\ \frac{1}{2} < s_0 < \frac{3}{2}$: massless vector ($s_0 = 1/2$), gravitino ($s_0 = 1$) and graviton ($s_0 = 3/2$) multiplets

$$SD\,(s_0 + 1, s_0|1) = D\,(s_0 + 1, s_0) \oplus D\,(s_0 + 3/2, s_0 + 1/2) \tag{2.4.29}$$

   The fields of these multiplets are all massless. Then, as we have seen, they tend with Inonü Wigner contraction to Poincaré massless fields. Actually, the entire multiplet tends to the corresponding massless multiplet of $\mathcal{N} = 1$ Poincaré supersymmetry, and has then the same structure.

3. $s_0 = 0,\ E_0 > \frac{1}{2}$: Wess Zumino multiplet

$$SD\,(E_0, 0|1) = D\,(E_0, 0) \oplus D\,(E_0 + 1/2, 1/2) \oplus D\,(E_0 + 1, 0)\,. \tag{2.4.30}$$

   When $E_0 = 1$, all the fields of this multiplet are massless. When $E_0 = 2$, some of the fields of this multiplet are massless. In the other cases, they are all massive.

4. $s_0 = 0,\ E_0 = \frac{1}{2}$: supersingleton representation

$$SD\,(1/2, 0|1) = D\,(1/2, 0) \oplus D\,(1, 1/2)\,. \tag{2.4.31}$$

   The $SO\,(3,2)$ UIRs of this $Osp\,(1|4)$ UIR are all singletons. Then, this representation does not have a realization as fields on the bulk, only on the boundary.

2.4.4 The structure of $\mathcal{N}=2$ supermultiplets

This case has been first studied in [23], where the list of the ${}^pK$ operators has been derived and some norms have been worked out, yielding the unitarity bounds, the shortening conditions and the structure of some multiplets. Then, in [14], the complete spectrum on a particular $\mathcal{N}=2$ supergravity compactification has been worked out (see chapter 3), and as a byproduct the remaining information on $\mathcal{N}=2$ UIRs has been found, namely, the absence of further unitarity bounds and shortening conditions, and the structure of all the multiplets.

The maximum number of fermionic generators allowed is $p=2\mathcal{N}=4$. The R–symmetry group is $SO(2)$, locally isomorphic to $U(1)$; the $U(1)$ UIRs are labeled by a rational number y usually called *hypercharge*, eigenvalue of the $U(1)$ generator Y. In the fermionic generators $\bar{a}^i_\alpha$ the index $i=1,2$ runs in the vector representation on $SO(2)$; the well–suited fermionic generators for the $U(1)$ form of the R–symmetry are

$$\begin{aligned} a^\pm_\alpha &= \frac{1}{\sqrt{2}}\left(a^1_\alpha \pm \mathrm{i} a^2_\alpha\right) \\ \overline{a}^\pm_\alpha &= \frac{1}{\sqrt{2}}\left(\overline{a}^1_\alpha \pm \mathrm{i}\overline{a}^2_\alpha\right), \end{aligned} \tag{2.4.32}$$

satisfying

$$\begin{aligned} \left(a^\pm_\alpha\right)^+ &= \bar{a}^\mp_\alpha \\ \left[H,\bar{a}^\pm_\alpha\right] &= \frac{1}{2}\bar{a}^\pm_\alpha \\ \left[Y,\bar{a}^\pm_\alpha\right] &= \pm\bar{a}^\pm_\alpha . \end{aligned} \tag{2.4.33}$$

Then, they are raising and lowering generators of hypercharge with weight 1.

In order to find all the operators ${}^pK$ we use the method previously described. We denote the operators by the representations of their indices; write all the representations allowed, and determine their $SO(2)\times U(1)$ labels. I remind that an $SO(2)$ UIR whose Young diagram (which is one row) has $n>0$ boxes, coincide to an $U(1)$ UIR with $y=n$ plus its conjugate representation, having $y=-n$; the $SO(2)$ singlet coincide to a real $U(1)$ UIR with $y=0$.

The complete list of ${}^pK$ operators is

	$SU(2)\times SU(2)$	$SU(2)\times SO(2)$ (${}^pK$ UIR)	(s,y) of ${}^pK$
B_0	$(1,1)$	$(1,1)$	$(0,0)$
B_1	$(\square,\square)$	$(\square,\square)$	$\left(\frac{1}{2},\pm 1\right)$
B_2	$(\square\square,1)$	$(\square\square,1)$	$(1,0)$
	$(1,\square\square)$	$(1,\square\square)\oplus(1,1)$	$(0,\pm 2)\oplus(0,0)$
B_3	$(\square,\square)$	$(\square,\square)$	$\left(\frac{1}{2},\pm 1\right)$
B_4	$(1,1)$	$(1,1)$	$(0,0)$

(2.4.34)

For example, the operator $(\square,\square)$ in B_3 is ${}^3K_\alpha \equiv \varepsilon^{\beta\gamma}\bar{a}^\pm_\alpha\bar{a}^+_\beta\bar{a}^-_\gamma$.

We can derive the complete list of states tensorizing the representations (2.4.34) with the quantum numbers of the possible vacua. In practice, the hypercharges adds up trivially, while the spins follow the usual rules for angular momentum composition. This

means that the number of states depends on the value of s_0, and the multiplets with spin not bigger than two have

$$0 \leq s_0 \leq 1. \tag{2.4.35}$$

Let us carry on the calculation of the norms. I do it only for the sectors B_1 and, partially, B_2: the other norm calculations are too long, and we search the missing information on unitarity bounds and shortening in other directions. We denote the vacuum, having norm 1, by

$$|\Omega\rangle \equiv |\,(E_0, s_0, y_0)\;\; E_0, s_0, m, y_0\rangle. \tag{2.4.36}$$

$\boxed{s = 0 \text{ case}}$

- B_1 sector
  The operators $\bar{a}_1^{\pm}$ give

$$\bar{a}_1^{\pm}|\Omega\rangle = R^{\pm}_{1/2}|\,(E_0+1/2, 1/2, y_0 \pm 1)\;\; E_0+1/2, 1/2, 1/2, y_0 \pm 1\rangle. \tag{2.4.37}$$

  We have

$$\langle\Omega|a_1^{\mp}\bar{a}_1^{\pm}|\Omega\rangle = E_0 \mp y_0 = \left|R^{\pm}_{1/2}\right|^2, \tag{2.4.38}$$

  that yields the condition $E_0 \mp y_0 \geq 0$.
  The unitarity condition arising from B_1 is then

$$E_0 \geq |y_0| \tag{2.4.39}$$

  The condition arising from $\bar{a}_2^{\pm}$ is identical. If it is strictly satisfied, the B_1 sector yields

$$SD\,(E_0+1/2, 1/2, y_0+1|2) \oplus SD\,(E_0+1/2, 1/2, y_0-1|2)\,. \tag{2.4.40}$$

  When the (2.4.39) is saturated,

  - if $y_0 > 0$
    $R^{+}_{1/2} = 0$ and the $SD\,(E_0+1/2, 1/2, y_0+1|2)$ field decouples;
  - if $y_0 < 0$
    $R^{-}_{1/2} = 0$ and the $SD\,(E_0+1/2, 1/2, y_0-1|2)$ field decouples.

  We will see that the case $y_0 = 0$ is excluded.

- B_2 sector
  The operators ${}^2K$ are

$$\begin{aligned}
\varepsilon^{\alpha\beta}\bar{a}^{+}_{\alpha}\bar{a}^{+}_{\beta} &= \left[\bar{a}^{+}_{1}, \bar{a}^{+}_{2}\right] && \text{with } s=0, y=2 \\
\varepsilon^{\alpha\beta}\bar{a}^{-}_{\alpha}\bar{a}^{-}_{\beta} &= \left[\bar{a}^{-}_{1}, \bar{a}^{-}_{2}\right] && \text{with } s=0, y=-2 \\
\varepsilon^{\alpha\beta}\bar{a}^{+}_{\alpha}\bar{a}^{-}_{\beta} &= \left[\bar{a}^{+}_{1}, \bar{a}^{-}_{2}\right] && \text{with } s=0, y=0 \\
\bar{a}^{+}_{(\alpha}\bar{a}^{-}_{\beta)} & && \text{with } s=1, y=0.
\end{aligned} \tag{2.4.41}$$

  The operator $\left[\bar{a}^{+}_{1}, \bar{a}^{+}_{2}\right]$ gives

$$\begin{aligned}
\left[\bar{a}^{+}_{1}, \bar{a}^{+}_{2}\right]|\Omega\rangle &= R^{+}_{0}|\,(E_0+1, 0, y_0+2)\, E_0+1, s_0, m, y_0+2\rangle + \\
&\quad +\beta M^{+}_{3}|\,(E_0, 0, y_0+2)\, E_0, 0, 0, y_0+2\rangle.
\end{aligned} \tag{2.4.42}$$

Let us determine β.

$$\begin{aligned}M_3^-\left[\bar{a}_1^+,\bar{a}_2^+\right]|\Omega\rangle &= \left[M_3^-\left[\bar{a}_1^+,\bar{a}_2^+\right]\right]|\Omega\rangle = 0 = \\ &= -2E_0\beta|\left(E_0,0,y_0+2\right)E_0,0,0,y_0+2\rangle,\end{aligned} \tag{2.4.43}$$

then $\beta = 0$. Now we can find $\left|R_0^+\right|^2$:

$$\langle\Omega|\left[a_2^-,a_1^-\right]\left[\bar{a}_1^+,\bar{a}_2^+\right]|\Omega\rangle = 4\left(E_0-y_0\right)\left(E_0-y_0-1\right) = \left|R_0^+\right|^2. \tag{2.4.44}$$

The operator $\left[\bar{a}_1^-,\bar{a}_2^-\right]$ is the complex conjugate of $\left[\bar{a}_1^+,\bar{a}_2^+\right]$, and as we have seen the conjugation of a $SO\left(2\right)\times U\left(1\right)$ changes the sign of the hypercharge. Then,

$$\left[\bar{a}_1^-,\bar{a}_2^-\right]|\Omega\rangle = R_0^-|\left(E_0+1,0,y_0-2\right)E_0+1,0,0,y_0-2\rangle \tag{2.4.45}$$

and

$$\left|R_0^-\right|^2 = 4\left(E_0+y_0\right)\left(E_0+y_0-1\right). \tag{2.4.46}$$

This yields a unitarity condition stronger than the (2.4.39). It is:

$$\begin{aligned}E_0 &\geq |y_0|+1 \\ &\text{or} \\ E_0 &= |y_0| \geq \frac{1}{2}.\end{aligned} \tag{2.4.47}$$

In fact, if $E_0 \neq |y_0|$ the (2.4.44) and (2.4.46) are both satisfied only if $E_0 \geq |y_0|+1$, while if $E_0 = y_0 > 0$ the (2.4.44) is zero, and the (2.4.46) gives the bound $2y_0-1 > 0$; the same thing happens if $E_0 = -y_0 > 0$. Notice that when $|y_0| < E_0 < |y_0|+1$ the unitarity bound is not satisfied; the set of allowed E_0, y_0 values is not connected.

When the (2.4.47) is strictly satisfied, the operators $\left[\bar{a}_1^+,\bar{a}_2^+\right]$, $\left[\bar{a}_1^-,\bar{a}_2^-\right]$ yield

$$D\left(E_0+1,0,y_0+2\right)\oplus\left(E_0+1,0,y_0-2\right). \tag{2.4.48}$$

When $E_0 = |y_0|+1$,

- if $y_0 > 0$
  $R_0^+ = 0$ and the $D\left(E_0+1,s_0,y_0+2\right)$ field decouples;
- if $y_0 < 0$
  $R_0^- = 0$ and the $D\left(E_0+1,s_0,y_0-2\right)$ field decouples;
- if $y_0 = 0$
  $R_0^+ = R_0^- = 0$ and the fields $D\left(E_0+1,s_0,y_0+2\right)$, $D\left(E_0+1,s_0,y_0-2\right)$ decouple.

When $E_0 = |y_0|$,

- if $y_0 > 1/2$
  $R_0^+ = 0$ and the $D\left(E_0+1,s_0,y_0+2\right)$ field decouples;
- if $y_0 < -1/2$
  $R_0^- = 0$ and the $D\left(E_0+1,s_0,y_0-2\right)$ field decouples;

– if $|y_0| = 1/2$
$R_0^+ = R_0^- = 0$ and the fields $D\,(E_0+1, s_0, y_0+2)$, $D\,(E_0+1, s_0, y_0-2)$ decouple.

I do not perform the calculation of the norms for the operators $\varepsilon^{\alpha\beta}\bar{a}_\alpha^+\bar{a}_\beta^-$, $\bar{a}_{(\alpha}^+\bar{a}_{\beta)}^-$ of the B_2 sector and for the operators in the B_3, B_4 sectors.

$$\boxed{s > 0 \text{ case}}$$

- $B1$ sector
The operators $\bar{a}_1^\pm$ give

$$\bar{a}_1^\pm|\Omega\rangle = R_{1/2}^\pm\sqrt{\frac{s_0+m+1}{2s_0+1}}|\,(E_0+1/2, s_0+1/2, y_0\pm1)\;E_0+1/2, s_0+1/2, m+1/2, y_0\pm1\rangle +$$
$$R_{-1/2}^\pm\sqrt{\frac{s_0-m}{2s_0+1}}|\,(E_0+1/2, s_0-1/2, y_0\pm1)\;E_0+1/2, s_0-1/2, m-1/2, y_0\pm1\rangle. \tag{2.4.49}$$

To find the values of the constants $R_{1/2}, R_{-1/2}$, we take specific values of m and determine the norms using the algebra (2.1.21).
Taking $m = s$,
$$\langle\Omega|a_1^\mp\bar{a}_1^\pm|\Omega\rangle = E_0 + s_0 \mp y_0 = \left|R_{1/2}^\pm\right|^2, \tag{2.4.50}$$
that yields the condition $E_0 + s_0 \mp y_0 \geq 0$.
Taking $m = -1/2$,

$$\langle\Omega|a_1^\mp\bar{a}_1^\pm|\Omega\rangle = E_0 - 1/2 \mp y_0 = \frac{1}{2}\left(\left|R_{1/2}\right|^2 + \left|R_{-1/2}\right|^2\right), \tag{2.4.51}$$

that yields the condition $E_0 - s_0 - 1 \mp y_0 \geq 0$.
The unitarity condition arising from B_1 is then

$$E_0 \geq s_0 + |y_0| + 1. \tag{2.4.52}$$

The condition arising from $\bar{a}_2^\pm$ is identical. This condition is coincident or stronger than the unitarity bound of AdS_4 fields for $s_0 > 1/2$. If it is strictly satisfied, the B_1 sector yields

$$D\,(E_0+1/2, s_0+1/2, y_0+1) \oplus D\,(E_0+1/2, s_0+1/2, y_0-1) \oplus$$
$$D\,(E_0+1/2, s_0-1/2, y_0+1) \oplus D\,(E_0+1/2, s_0-1/2, y_0-1)\,. \tag{2.4.53}$$

When the (2.4.52) is saturated,

– if $y_0 > 0$
$R_{-1/2}^+ = 0$ and the $D\,(E_0+1/2, s_0-1/2, y_0+1)$ field decouples;
– if $y_0 < 0$
$R_{-1/2}^- = 0$ and the $D\,(E_0+1/2, s_0-1/2, y_0-1)$ field decouples;

– if $y_0 = 0$
$R^+_{-1/2} = R^-_{-1/2} = 0$ and the fields $D\,(E_0{+}1/2, s_0{-}1/2, y_0{+}1)$, $D\,(E_0{+}1/2, s_0{-}1/2, y_0{-}1)$ decouple; furthermore, in this case the fields $D\,(E_0, s_0, \pm 1)$ and $D\,(E_0{+}1/2, s_0{+}1/2, \pm 1)$ are massless.

- $B2$ sector
The operators ${}^2K$ are

$$\begin{array}{rcll} \varepsilon^{\alpha\beta}\bar{a}^+_\alpha \bar{a}^+_\beta &=& \left[\bar{a}^+_1, \bar{a}^+_2\right] & \text{with } s=0, y=2 \\ \varepsilon^{\alpha\beta}\bar{a}^-_\alpha \bar{a}^-_\beta &=& \left[\bar{a}^-_1, \bar{a}^-_2\right] & \text{with } s=0, y=-2 \\ \varepsilon^{\alpha\beta}\bar{a}^+_\alpha \bar{a}^-_\beta &=& \left[\bar{a}^+_1, \bar{a}^-_2\right] & \text{with } s=0, y=0 \\ \bar{a}^+_{(\alpha}\bar{a}^-_{\beta)} & & & \text{with } s=1, y=0. \end{array} \tag{2.4.54}$$

The operator $\left[\bar{a}^+_1, \bar{a}^+_2\right]$ gives

$$\begin{aligned} \left[\bar{a}^+_1, \bar{a}^+_2\right]|\Omega\rangle &= R^+_0 |\,(E_0+1, s_0, y_0+2)\, E_0+1, s_0, m, y_0+2\rangle + \\ &\quad + \text{boosted elements of } D\,(E_0, s_0, y_0+2)\,. \end{aligned} \tag{2.4.55}$$

In order to calculate $\left|R^+_0\right|^2$ we take $m = s_0$. Then

$$\begin{aligned} \left[\bar{a}^+_1, \bar{a}^+_2\right]|\Omega\rangle &= R^+_0 |\,(E_0+1, s_0, y_0+2)\, E_0+1, s_0, s_0, y_0+2\rangle + \\ &\quad + \alpha^+ M^+_{(+)} |\,(E_0, s_0, y_0+2)\, E_0, s_0, s_0-1, y_0+2\rangle + \\ &\quad + \beta^+ M^+_3 |\,(E_0, s_0, y_0+2)\, E_0, s_0, s_0, y_0+2\rangle\,. \end{aligned} \tag{2.4.56}$$

Let us determine α^+ and β^+. By means of the algebra (2.1.21) we find

$$\begin{aligned} & M^-_{(-)}\left[\bar{a}^+_1, \bar{a}^+_2\right]|\Omega\rangle = \left[M^-_{(-)}\left[\bar{a}^+_1, \bar{a}^+_2\right]\right]|\Omega\rangle = 0 = \\ & = \left(-2\,(E_0+s_0-1)\,\alpha^+ + 2\sqrt{s_0}\beta^+\right) |\,(E_0, s_0, y_0+2)\, E_0, s_0, s_0-1, y_0+2\rangle \end{aligned} \tag{2.4.57}$$

$$\begin{aligned} & M^-_3\left[\bar{a}^+_1, \bar{a}^+_2\right]|\Omega\rangle = \left[M^-_3\left[\bar{a}^+_1, \bar{a}^+_2\right]\right]|\Omega\rangle = 0 = \\ & = \left(2\sqrt{s_0}\alpha^+ - 2E_0\beta^+\right) |\,(E_0, s_0, y_0+2)\, E_0, s_0, s_0, y_0+2\rangle, \end{aligned} \tag{2.4.58}$$

then

$$\alpha^+ = \beta^+ = 0. \tag{2.4.59}$$

Now we can find $\left|R^+_0\right|^2$:

$$\begin{aligned} \left|R^+_0\right|^2 &= \langle\Omega|\left[a^-_2, a^-_1\right]\left[\bar{a}^+_1, \bar{a}^+_2\right]|\Omega\rangle = ((E_0{+}s_0{-}1{-}y_0)\,(E_0{-}s_0{-}y_0){-}2s_0)+ \\ &\quad -(-(E_0{-}s_0{-}y_0)\,(E_0{+}s_0{-}y_0{-}1){+}2s_0)+ \\ &- \;(-(E_0{+}s_0{-}y_0)\,(E_0{-}s_0{-}y_0{-}1)){+}((E_0{+}s_0{-}y_0)\,(E_0{-}s_0{-}y_0{-}1)) = \\ &= 4\,(E_0{-}y_0{+}s_0)\,(E_0{-}y_0{-}s_0{-}1)\;. \end{aligned} \tag{2.4.60}$$

The operator $\left[\bar{a}^-_1, \bar{a}^-_2\right]$ is the complex conjugate of $\left[\bar{a}^+_1, \bar{a}^+_2\right]$, and as we have seen the conjugation of a $U\,(1)$ representation changes the sign of the hypercharge. Then,

$$\left[\bar{a}^-_1, \bar{a}^-_2\right]|\Omega\rangle = R^-_0 |\,(E_0+1, s_0, y_0-2)\, E_0+1, s_0, m, y_0-2\rangle \tag{2.4.61}$$

and

$$\left|R_0^-\right|^2 = 4\left(E_0 + y_0 + s_0\right)\left(E_0 + y_0 - s_0 - 1\right). \tag{2.4.62}$$

The condition (2.4.52)

$$E_0 \geq s_0 + |y_0| + 1 \tag{2.4.63}$$

guarantees $\left|R_0^+\right|^2 \geq 0$ and $\left|R_0^-\right|^2 \geq 0$. If it is strictly satisfied, the operators $\left[\bar{a}_1^+, \bar{a}_2^+\right]$, $\left[\bar{a}_1^-, \bar{a}_2^-\right]$ yield

$$D\left(E_0 + 1, s_0, y_0 + 2\right) \oplus \left(E_0 + 1, s_0, y_0 - 2\right). \tag{2.4.64}$$

When it is saturated,

- if $y_0 > 0$
  $R_0^+ = 0$ and the $D\left(E_0 + 1, s_0, y_0 + 2\right)$ field decouples;
- if $y_0 < 0$
  $R_0^- = 0$ and the $D\left(E_0 + 1, s_0, y_0 - 2\right)$ field decouples;
- if $y_0 = 0$
  $R_0^+ = R_0^- = 0$ and the fields $D\left(E_0 + 1, s_0, y_0 + 2\right)$, $D\left(E_0 + 1, s_0, y_0 - 2\right)$ decouple.

We do not perform the calculation of the norms for the operators $\varepsilon^{\alpha\beta}\bar{a}_\alpha^+\bar{a}_\beta^-$, $\bar{a}_{(\alpha}^+\bar{a}_{\beta)}^-$ in the B_2 sector and for the operators in the B_3, B_4 sectors.

At this point we do not know if there are other unitarity bounds and shortening conditions in addition to the (2.4.47) for $s_0 = 0$ and (2.4.52) for $s_0 > 0$. Furthermore, we do not know which other fields decouple in the shortened representation just found, namely, the complete structure of the short $\mathcal{N} = 2$ multiplets. We know from the literature on supergravity the complete structure of the massless supermultiplets, but not of the massive ones. A possible way to get this information is by deriving the norms of the remainings states, but it would be a very lengthy calculation. I prefer to utilize the tricks listed in page 2.4.2.

First of all, we can use the results of harmonic analysis on the M^{111} manifold (and, in part, Q^{111}), described in next chapter, which gives the complete mass spectrum on the corresponding Kaluza Klein supergravity solutions (which have, in both cases, $\mathcal{N} = 2$ supersymmetry). We have found the masses, energies and hypercharges (and flavour quantum numbers, which however are not relevant in the present context) of almost all the particles of these supergravity; this is enough to organize them in supermultiplets, and by means of the part just derived on supermultiplet structure and of the decomposition under $\mathcal{N} = 1$ supermultiplets we can complete the spectrum; as a byproduct, we find the complete structure of the multiplets appearing in these supergravities, related with their energies and hypercharges. This confirms that the only unitarity bounds in $\mathcal{N} = 2$ supersymmetry are

$$\begin{gathered} E_0 \geq s_0 + |y_0| + 1 \qquad s_0 \geq 0 \\ \text{or} \\ E_0 = |y_0| \geq \frac{1}{2} \qquad s_0 = 0, \end{gathered} \tag{2.4.65}$$

and the only shortening conditions are the ones we found, corresponding to the saturations of these bounds. This procedure is explained in the next chapter, here I report the results.

I give the list of the $Osp\,(2|4)$ UIRs with maximal spin not greater than two. Their explicit structures are showed in tables 2.1,. . .,2.11. I remind that since we are considering the R–symmetry $SO\,(2)$ in the complex form $U\,(1)$, when the hypercharge is different from zero, the supermultiplet is complex; there are then two supermultiplets, conjugate each other with opposite values of y_0; in this case, I display in the tables the one with $y_0 > 0$. When, on the contrary, $y_0 = 0$, the supermultiplet is real, and then there is only one of them.

1. $E_0 > s_0 + |y_0| + 1,\ \ 0 < s_0 < 1$: *long multiplets*; long graviton multiplet ($s_0 = 1$), long gravitino multiplet ($s_0 = 1/2$), long vector multiplet ($s_0 = 0$). They have the structures displayed in tables 2.1,2.2,2.3. The fields of these multiplets are all massive. Their decomposition under $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ is, if $y_0 \neq 0$,

$$\begin{aligned} SD\,(E_0, 1, y_0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 3/2|1) \oplus SD\,(E_0 + 1, 1|1) \oplus \\ & SD\,(E_0 + 1/2, 1/2|1) \oplus \oplus SD\,(E_0, 1|1) \oplus \\ & \oplus SD\,(E_0 + 1/2, 3/2|1) \oplus SD\,(E_0 + 1, 1|1) \oplus \\ & SD\,(E_0 + 1/2, 1/2|1) \oplus SD\,(E_0, 1|1) \end{aligned} \tag{2.4.66}$$

$$\begin{aligned} SD\,(E_0, 1/2, y_0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 1|1) \oplus SD\,(E_0 + 1, 1/2|1) \oplus \\ & SD\,(E_0, 1/2|1) \oplus SD\,(E_0 + 1/2, 0|1) \oplus \\ & \oplus SD\,(E_0 + 1/2, 1|1) \oplus SD\,(E_0 + 1, 1/2|1) \oplus \\ & SD\,(E_0, 1/2|1) \oplus \oplus SD\,(E_0 + 1/2, 0|1) \end{aligned} \tag{2.4.67}$$

$$\begin{aligned} SD\,(E_0, 0, y_0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 1/2|1) \oplus SD\,(E_0 + 1, 0|1) \oplus \\ & SD\,(E_0, 0|1) \oplus \\ & \oplus SD\,(E_0 + 1/2, 1/2|1) \oplus SD\,(E_0 + 1, 0|1) \oplus \\ & SD\,(E_0, 0|1) \end{aligned} \tag{2.4.68}$$

while if $y_0 = 0$ it is

$$\begin{aligned} SD\,(E_0, 1, 0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 3/2|1) \oplus SD\,(E_0 + 1, 1|1) \oplus \\ & SD\,(E_0 + 1/2, 1/2|1) \oplus \oplus SD\,(E_0, 1|1) \end{aligned} \tag{2.4.69}$$

$$\begin{aligned} SD\,(E_0, 1/2, 0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 1|1) \oplus SD\,(E_0 + 1, 1/2|1) \oplus \\ & SD\,(E_0, 1/2|1) \oplus SD\,(E_0 + 1/2, 0|1) \end{aligned} \tag{2.4.70}$$

$$\begin{aligned} SD\,(E_0, 0, 0|2) \quad \longrightarrow \quad & SD\,(E_0 + 1/2, 1/2|1) \oplus SD\,(E_0 + 1, 0|1) \oplus \\ & SD\,(E_0, 0|1)\,. \end{aligned} \tag{2.4.71}$$

2. $E_0 = s_0 + |y_0| + 1,\ \ |y_0| > 0,\ \ 0 < s_0 < 1$: *short multiplets*; short graviton multiplet ($s_0 = 1$), short gravitino multiplet ($s_0 = 1/2$), short vector multiplet ($s_0 = 0$). They have the structures displayed in tables 2.4,2.5,2.6. The fields of these multiplets are all massive. Their decomposition under $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ is

$$SD\,(|y_0| + 2, 1, y_0|2) \quad \longrightarrow \quad SD\,(|y_0| + 5/2, 3/2|1) \oplus SD\,(|y_0| + 2, 1|1) \oplus$$

$$
\begin{aligned}
& \oplus SD\,(|y_0|+5/2,3/2|1) \oplus SD\,(|y_0|+2,1|1) && (2.4.72)\\
SD\,(|y_0|+3/2,1/2,y_0|2) \;\longrightarrow\; & SD\,(|y_0|+2,1|1) \oplus SD\,(|y_0|+3/2,1/2|1) \oplus \\
& \oplus SD\,(|y_0|+2,1|1) \oplus SD\,(|y_0|+3/2,1/2|1) && (2.4.73)\\
SD\,(|y_0|+1,0,y_0|2) \;\longrightarrow\; & SD\,(|y_0|+3/2,1/2|1) \oplus SD\,(|y_0|+1,0|1) \oplus \\
& \oplus SD\,(|y_0|+3/2,1/2|1) \oplus SD\,(|y_0|+1,0|1)\;. && (2.4.74)
\end{aligned}
$$

3. $E_0 = s_0 + 1,\ y_0 = 0,\ 0 < s_0 < 1$: *massless multiplets*; massless graviton multiplet ($s_0 = 1$), massless gravitino multiplet ($s_0 = 1/2$), massless vector multiplet ($s_0 = 0$). They have the structures displayed in tables 2.8,2.9,2.10. The fields of these multiplets are all massless. Their decomposition under $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ is

$$SD\,(2,1,0|2) \;\longrightarrow\; SD\,(5/2,3/2|1) \oplus SD\,(2,1|1) \tag{2.4.75}$$
$$SD\,(3/2,1/2,0|2) \;\longrightarrow\; SD\,(2,1|1) \oplus SD\,(3/2,1/2|1) \tag{2.4.76}$$
$$SD\,(1,0,0|2) \;\longrightarrow\; SD\,(3/2,1/2|1) \oplus SD\,(1,0|1)\,. \tag{2.4.77}$$

4. $E_0 = |y_0| > 1/2,\ s_0 = 0$: *hypermultiplet*. It has the structure displayed in table 2.7. For $E_0 \neq 1,2$ the fields of this multiplet are all massive. For $E_0 = 2$, some of them are massless, some other massive, and for $E_0 = 1$ all the fields of the multiplet are massless. However, this multiplet is always complex, because $y_0 \neq 0$. Its decomposition under $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ is

$$SD\,(|y_0|,0,y_0|2) \longrightarrow SD\,(|y_0|,0|1) \oplus SD\,(|y_0|,0|1)\,. \tag{2.4.78}$$

   Notice that this multiplet arises from a particular shortening (the one with $E_0 = |y_0|$, different from the one with $E_0 = |y_0| + 1$) of the vector multiplet, under which also the maximal spin state, the vector, decouple. So the multiplet does not contain spins greater than 1/2; this phenomenom is not possible in long multiplets with $\mathcal{N} = 2$ supersymmetry, due to the existence of an operator in the enveloping algebra having spin one (see (2.4.34)). For historical reasons, multiplets with spin not greater than 1/2 are called hypermultiplets.

5. $E_0 = |y_0| = 1/2$: *supersingleton representation*. It has the structure displayed in table 2.11. The $SO\,(3,2)$ UIRs of this $Osp\,(2|4)$ UIR are all singletons; then, this representation is not a multiplet of supergravity fields, it does not have a realization as fields on the bulk, but only on the boundary. Its decomposition under $\mathcal{N} = 2 \longrightarrow \mathcal{N} = 1$ is

$$SD\,(1/2,0,1/2|2) \longrightarrow SD\,(1/2,0|1) \oplus SD\,(1/2,0|1)\,. \tag{2.4.79}$$

To be precise, in principle there could be other unitarity bounds or shortening phenomena arising from the norms not evaluated, but this seems very unlikely because the Kaluza Klein analysis of two spectra does not show anything of that.

2.4.5 The structure of $\mathcal{N}=3$ supermultiplets

This case has been first studied in [24], where some norms have been worked out giving the unitarity bounds and some shortening conditions, and the structure of the short vector multiplet has been worked out. Then, in [17], the list of the ${}^{p}K$ operators and the $\mathcal{N}=3 \longrightarrow \mathcal{N}=2$ decompositions have been derived, relying on the results of [53], and the complete spectrum of a particular $\mathcal{N}=3$ supergravity compactifiation has been worked out (see chapter 3); as a byproduct, the lacking information on $\mathcal{N}=3$ UIRs has been found, namely, the absence of further unitarity bounds, the remaining shortening conditions, and the structure of all the multiplets (with the exception of the ones with $J_0 = 3/2, 1/2$, not appearing in the spectrum of our compactification).

The maximum number of fermionic generators allowed is $p = 2\mathcal{N} = 8$. The R–symmetry group is $SO(3)$, locally isomorphic to $SU(2)$, which is the form we consider. An R–symmetry UIR is labeled by its $SU(2)_R$ spin, which we call *isospin* J, and its states are labeled by the third isospin component $M \in [-J, J]$. In the fermionic generators $\bar{a}^i_\alpha$ the index $i = 1, \ldots, 3$ runs in the vector representation of $SO(3)$; the well–suited fermionic generators for the $SU(2)$ form of the R–symmetry are

$$\begin{aligned} a^{\pm}_\alpha &= \frac{1}{\sqrt{2}}\left(a^1_\alpha \pm a^2_\alpha\right) \\ a^3_\alpha & \end{aligned} \tag{2.4.80}$$

satisfying

$$\begin{aligned} \left(a^{\pm}_\alpha\right)^+ &= \bar{a}^{\mp}_\alpha \\ \left[H, \bar{a}^{\pm,3}_\alpha\right] &= \frac{1}{2}\bar{a}^{\pm,3}_\alpha \\ \left[M, \bar{a}^{\pm}_\alpha\right] &= \pm\bar{a}^{\pm}_\alpha \\ \left[M, \bar{a}^{3}_\alpha\right] &= 0. \end{aligned} \tag{2.4.81}$$

Let us find all the operators ${}^{p}K$. As usual, we denote the operators by the representations of their indices; write all the representations allowed, and determine their $SO(2) \times SU(2)$ labels. I remind that for $SU(3)$ representations $\begin{smallmatrix}\square\\\square\end{smallmatrix} \simeq \square$. Furthermore I remind that, under the isomorphism $SO(3) \simeq SU(2)$, the simplest UIRs transforms as

$$\begin{array}{ccc} SO(3) & SU(2) & \\ 1 & 1 & J=0 \\ \square & \square\square & J=1 \\ \square\square & \square\square\square\square & J=2\,. \end{array} \tag{2.4.82}$$

For example, $\bar{a}^i_\alpha$ are in the $\mathbf{3}$ of $SO(3) \simeq SU(2)$, that is the $\square$ of $SO(3)$ and the $\square\square$ of $SU(2)$.

The complete list of ${}^pK$ operators is

	$SU(2)\times SU(3)$	$SU(2)\times SU(2)$ (${}^pK$ UIR)	(s,J) of ${}^pK$
B_0	$(1,1)$	$(1,1)$	$(0,0)$
B_1	$(\square,\square)$	$(\square,\square\square)$	$\left(\frac{1}{2},1\right)$
B_2	$(\square\square,\square)$	$(\square\square,\square\square)$	$(1,1)$
	$(1,\square\square)$	$(1,\square\square\square\square)\oplus(1,1)$	$(0,2)\oplus(0,0)$
B_3	$\left(\square,\begin{smallmatrix}\square\square\\ \square\end{smallmatrix}\right)$	$(\square,\square\square\square\square)\oplus(\square,\square\square)$	$\left(\frac{1}{2},2\right)\oplus\left(\frac{1}{2},1\right)$
	$(\square\square\square,1)$	$(\square\square\square,1)$	$\left(\frac{3}{2},0\right)$
B_4	$(1,\square\square)$	$(1,\square\square\square\square)\oplus(1,1)$	$(0,2)\oplus(0,0)$
	$(\square\square,\square)$	$(\square\square,\square\square)$	$(1,1)$
B_5	$(\square,\square)$	$(\square,\square\square)$	$\left(\frac{1}{2},1\right)$
B_6	$(1,1)$	$(1,1)$	$(0,0)$

(2.4.83)

We can derive the complete list of states tensorizing the representations (2.4.83) with the quantum numbers of the possible vacua. Both the spins and the isospins follow the usual rules of angular momentum composition. This means that the number of states depends on the value of s_0, and the multiplets with spin not bigger than two have

$$0\le s_0\le\frac{1}{2}. \tag{2.4.84}$$

The norms of the states created by the sector B_1 have been derived in [24]. I give here their results, without repeating their proof. The highest weight operator $\bar{a}_1^+$ yields

$$\begin{aligned}\bar{a}_1^+|\,(E_0,s_0,J_0)\,E_0,s_0,m,J_0,M\rangle &= \sum_{\mu\nu}R_{\mu\nu}\langle s_0+\mu,m+\frac{1}{2}|s_0,m,\frac{1}{2},\frac{1}{2}\rangle\,\cdot\\ \cdot\langle J_0+\nu,M+1|J_0,M,1,1\rangle &\left|\left(E_0+\frac{1}{2},s_0+\mu,J_0+\nu\right)E_0+\frac{1}{2},s_0+\mu,m+\frac{1}{2},M+1\right\rangle.\end{aligned} \tag{2.4.85}$$

By giving appropriate values to m,M, one finds the expression of the $|R_{\mu\nu}|$:

$$\begin{aligned}\left|R_{\frac{1}{2},1}\right|^2 &= E_0+s_0-J_0,\\ \left|R_{-\frac{1}{2},1}\right|^2 &= E_0-s_0-J_0-1,\\ \left|R_{\frac{1}{2},0}\right|^2 &= E_0+s_0+1,\\ \left|R_{-\frac{1}{2},0}\right|^2 &= E_0-s_0,\\ \left|R_{\frac{1}{2},-1}\right|^2 &= E_0+s_0+J_0+1,\\ \left|R_{-\frac{1}{2},-1}\right|^2 &= E_0-s_0+J_0.\end{aligned} \tag{2.4.86}$$

I remind that when $s_0=0$ the Clebsch Gordan coefficients multiplying $R_{-\frac{1}{2},\nu}$ in the expansion (2.4.85) vanish.

These norms yield the following unitarity bounds:

$$\begin{aligned} E_0 \geq s_0 + J_0 + 1 \qquad & s_0 > 0 \\ E_0 \geq J_0 \qquad & s_0 = 0. \end{aligned} \tag{2.4.87}$$

From the norm evaluation of some operators in other sectors done in [24] another unitarity bound arises: when $s_0 = 0$, we can have $E_0 = J_0$ or $E_0 > J_0+1$, but not $J_0 < E_0 < J_0+1$; as in the $\mathcal{N} = 2$ case, there is a "disconnected" unitarity condition. Furthermore, as in the $\mathcal{N} = 2$ case, in the corresponding short multiplet also the maximal spin field decouple, yielding a vector multiplet instead of a gravitino multiplet. The structure of this supermultiplet has been completely determined in [24].

Using the results of the harmonic analysis on the N^{010} manifold given in [53], we have found in [17] the complete spectrum (as described in the next chapter) of the corresponding Kaluza Klein solution, which has $\mathcal{N} = 3$ supersymmetry. We have found the masses, energies and isospins of almost all the particles of this supergravity; this is enough to organize them in supermultiplets, and by means of the results of [24] and of the decomposition under $\mathcal{N} = 2$ supermultiplets we can complete the spectrum; as a byproduct, we found the complete structure of all the multiplets appearing in this supergravity, related with their energies and isospins. This confirms that the only unitarity bounds in $\mathcal{N} = 3$ supersymmetry are

$$\begin{array}{llll} E_0 & \geq s_0 + J_0 + 1 & & s_0 \geq 0 \\ \text{or} & & & \\ E_0 & = J_0 & & s_0 = 0. \end{array} \tag{2.4.88}$$

We have short representations when

$$E_0 = J_0 + s_0 + 1 \;\; \text{or} \;\; E_0 = J_0, \;\; s_0 = 0. \tag{2.4.89}$$

I stress that there is another shortening mechanism of a completely different origin. The creation operators that act on the vacuum have isospin $0 \leq J_0 \leq 2$. If the isospin of the vacuum is $J_0 \geq 2$, the creation operators give rise to states with isospin in the range $J_0 - J \leq J_{\text{composite}} \leq J_0 + J$. Yet, in the case where $0 \leq J_0 < 2$, some of these states cannot appear. This mechanism is not related to an unitarity bound, and these representations are not BPS states of supergravity, nor primary conformal operators on the boundary. Then we call *long* the representations with $E_0 > s_0 + J_0 + 1$, even if $0 \leq J_0 < 2$. In this context, the massless representations are the short ones with $J_0 = 0$ for the case of the massless graviton and gravitino multiplets and with $J_0 = 1$ for the case of the massless vector multiplets; the supersingleton representation is the short vector multiplet with $J_0 = 1/2$, $SD\,(1/2, 0, 1/2|3)$. Unfortunately, only states with integer isospins appear in the N^{010} spectrum, then we do not have enough information to know the complete structure of multiplets with $J_0 = 3/2$ and $J_0 = 1/2$, with the exception of the supersingleton which we worked out a part. However, the $\mathcal{N} = 2$ decomposition we found is true for all the values of J_0.

The complete list of the $Osp\,(3|4)$ UIRs with $s_{max} \leq 2$ is given below:

- *long graviton multiplet* $SD\,(E_0, 1/2, J_0|3)$ where $E_0 > J_0 + 3/2$, see table 2.12;
- *long gravitino multiplet* $SD\,(E_0, 0, J_0|3)$ where $E_0 > J_0 + 1$, see table 2.13;

- *short graviton multiplet* $SD\,(J_0+3/2, 1/2, J_0|3)$, see table 2.14;
- *short gravitino multiplet* $SD\,(J_0+1, 0, J_0|3)$, see table 2.15;
- *short vector multiplet* $SD\,(J_0, 0, J_0|3),\ J_0 \geq 1$, see table 2.16;
- *supersingleton representation* $SD\,(1/2, 0, 1/2|3)$, see table 2.16.

Note that there are no long vector multiplets, and no hypermultiplets at all.

The $\mathcal{N}=3 \longrightarrow \mathcal{N}=2$ decompositions of the above multiplets are listed below. I remind that an $Osp\,(2|4)$ UIR is denoted by $SD\,(E_0, s_0, y_0|2)$, and if $y_0 \neq 0$ this is a complex representation, the conjugate one having opposite hypercharge. So, in the following list when there is a complex representation we write $SD\,(E_0, s_0, y_0|2) \oplus SD\,(E_0, s_0, -y_0|2)$. Then, for example, the $\mathcal{N}=3$ supersingleton representation with this convention decomposes $SD\,(1/2, 0, 1/2|3) \longrightarrow SD\,(1/2, 0, 1/2|2) \oplus SD\,(1/2, 0, -1/2|2)$, but actually it coincide with the $\mathcal{N}=2$ supersingleton representation.

$$
\begin{aligned}
SD\,(E_0, 1/2, J_0|3) \;\longrightarrow\;& \bigoplus_{y=-J_0}^{J_0} SD\,(E_0+1/2, 1, y|2) \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(E_0, 1/2, y|2) \\
& \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(E_0+1, 1/2, y|2) \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(E_0+1/2, 0, y|2) \\
& \text{where } E_0 > J_0+3/2 \\
SD\,(J_0+3/2, 1/2, J_0|3) \;\longrightarrow\;& \bigoplus_{y=-J_0}^{J_0} SD\,(J_0+2, 1, y|2) \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(J_0+3/2, 1/2, y|2) \\
SD\,(E_0, 0, J_0|3) \;\longrightarrow\;& \bigoplus_{y=-J_0}^{J_0} SD\,(E_0+1/2, 1/2, y|2) \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(E_0+1, 0, y|2) \\
& \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(E_0, 0, y|2) \qquad \text{where } E_0 > J_0+1 \\
SD\,(J_0+1, 0, J_0|3) \;\longrightarrow\;& \bigoplus_{y=-J_0}^{J_0} SD\,(J_0+3/2, 1/2, y|2) \oplus \bigoplus_{y=-J_0}^{J_0} SD\,(J_0+1, 0, y|2) \\
SD\,(J_0, 0, J_0|3) \;\longrightarrow\;& \bigoplus_{y=-J_0}^{J_0} SD\,(J_0, 0, y|2)\ .
\end{aligned}
\tag{2.4.90}
$$

Notice that while $Osp\,(3|4) \supset Osp\,(2|4)$, $Osp\,(3|4) \not\supset Osp\,(1|4) \times SO\,(3)$. It is then impossible in general to decompose the $\mathcal{N}=3$ UIRs in $\mathcal{N}=1$ UIRs with definite isospin.

2.5 The AdS_4 and ∂AdS_4 superspaces

As I said, the anti–de Sitter superspace is the following supercoset:

$$
AdS_{4|\mathcal{N}} \equiv \frac{Osp(\mathcal{N}|4)}{SO(1,3) \times SO(\mathcal{N})} \tag{2.5.1}
$$

and has 4 bosonic coordinates labelling the points in AdS_4 and $4 \times \mathcal{N}$ fermionic coordinates $\Theta^{\alpha i}$ that transform as Majorana spinors under $SO(1,3)$ and as vectors under $SO(\mathcal{N})$. There are many possible coordinate choices for parametrizing such a manifold, but as far as the bosonic submanifold is concerned it was shown in [41] that a particularly useful parametrization is the solvable one where the AdS_4 coset is regarded as a *non–compact solvable group manifold*:

$$AdS_4 \equiv \frac{SO(2,3)}{SO(1,3)} = \exp\left[Solv_{adS}\right] . \tag{2.5.2}$$

The solvable algebra $Solv_{adS}$ is spanned by the unique non–compact Cartan generator D belonging to the coset and by three abelian operators P_m ($m = 0,1,2$) generating the translation subalgebra in $d = 1+2$ dimensions. The solvable coordinates are

$$\rho \quad \leftrightarrow \quad D \quad ; \quad z^m \quad \leftrightarrow \quad P_m \tag{2.5.3}$$

and in such coordinates the AdS_4 metric takes the form [4]

$$\rho^2 \left(-dz_0^2 + dz_1^2 + dz_2^2\right) + \frac{1}{\rho^2} d\rho^2 . \tag{2.5.4}$$

Hence ρ is interpreted as measuring the distance from the brane–stack and z^m are interpreted as cartesian coordinates on the brane boundary $\partial(AdS_4)$. A possible question is: can such a solvable parametrization of AdS_4 be extended to a supersolvable parametrization of anti–de Sitter superspace as defined in (2.5.1)? In practice that means to single out a solvable superalgebra with 4 bosonic and $4 \times \mathcal{N}$ fermionic generators. As shown in [54], this turns out to be impossible, yet there is a supersolvable algebra $Ssolv_{adS}$ with 4 bosonic and $2 \times \mathcal{N}$ fermionic generators whose exponential defines the *solvable anti–de Sitter superspace*:

$$AdS_{4|2\mathcal{N}}^{(Solv)} \equiv \exp\left[Ssolv_{adS}\right] . \tag{2.5.5}$$

The supermanifold (2.5.5) is also a supercoset of the same supergroup $Osp(\mathcal{N}|4)$ but with respect to a different subgroup:

$$AdS_{4|2\mathcal{N}}^{(Solv)} = \frac{Osp(4|\mathcal{N})}{CSO(1,2|\mathcal{N})} \tag{2.5.6}$$

where $CSO(1,2|\mathcal{N}) \subset Osp(\mathcal{N}|4)$ is generated by an algebra containing $3 + 3 + \frac{\mathcal{N}(\mathcal{N}-1)}{2}$ bosonic generators and $2 \times \mathcal{N}$ fermionic ones. This algebra is the semidirect product:

$$cso(1,2|\mathcal{N}) \quad = \quad \underbrace{iso(1,2|\mathcal{N}) \oplus so(\mathcal{N})}_{\text{semidirect}} \tag{2.5.7}$$

of the $\mathcal{N}$–extended *superPoincaré* algebra in three dimensions $iso(1,2|\mathcal{N})$ with $so(\mathcal{N})$. It should be clearly distinguished from the central extension of the Poincaré superalgebra, $Z[iso(1,2|\mathcal{N})]$, which has the same number of generators but different commutation relations. Indeed there are three essential differences that it is worth to recall at this point:

[4] which is the form appearing in (1.1.28), where ρ is called U

1. In $Z\left[ISO(1,2|\mathcal{N})\right]$ the $\mathcal{N}(\mathcal{N}-1)/2$ internal generators Z^{ij} are abelian, while in $CSO(1,2|\mathcal{N})$ the corresponding T^{ij} are non abelian and generate $SO(\mathcal{N})$.
2. In $Z\left[ISO(1,2|\mathcal{N})\right]$ the supercharges $q^{\alpha i}$ commute with Z^{ij} (these are in fact central charges), while in $CSO(1,2|\mathcal{N})$ they transform as vectors under T^{ij}.
3. In $Z\left[ISO(1,2|\mathcal{N})\right]$ the anticommutator of two supercharges yields, besides the translation generators P_m, also the central charges Z^{ij}, while in $CSO(1,2|\mathcal{N})$ this is not true.

In both cases of fig.2.1 and fig.2.2 if one takes the subset of generators of positive grading plus the abelian grading generator $X = \left\{ \begin{array}{l} E \\ D \end{array} \right.$ one obtains a *solvable superalgebra* of dimension $4+2\mathcal{N}$. It is however only in the non compact case of fig.2.2 that the bosonic subalgebra of the solvable superalgebra generates anti–de Sitter space AdS_4 as a solvable group manifold.

The structure of $ISO(1,2|\mathcal{N}) \subset Osp(\mathcal{N}|4)$ can be easily seen in picture 2.2, displaying the root diagram of the superconformal interpretation of the $Osp(\mathcal{N}|4)$. $CSO(1,2|\mathcal{N})$ is spanned by the generators out of the square, which have null or negative grading, namely the conformal boosts K_m, the Lorentz generators J_m and the special conformal supersymmetries s^i_α. Notice that the generators in the square define a solvable subalgebra at sight, because in a root diagram the commutator of two generators, if not zero, corresponds to the vector sum of the vectors corresponding to the two generators. The solvable superalgebra $Ssolv_{adS}$ mentioned in eq. (2.5.5) is the vector span of the following generators:

$$Ssolv_{adS} \equiv \mathrm{span}\left\{P_m, D, q^{\alpha i}\right\}. \tag{2.5.8}$$

Being a coset, the solvable AdS–superspace $AdS^{(Solv)}_{4|2\mathcal{N}}$ supports a non linear representation of the full $Osp(\mathcal{N}|4)$ superalgebra. As shown in [54], we can regard $AdS^{(Solv)}_{4|2\mathcal{N}}$ as ordinary anti–de Sitter superspace $AdS_{4|\mathcal{N}}$ where $2\times\mathcal{N}$ fermionic coordinates have being eliminated by fixing κ–supersymmetry.

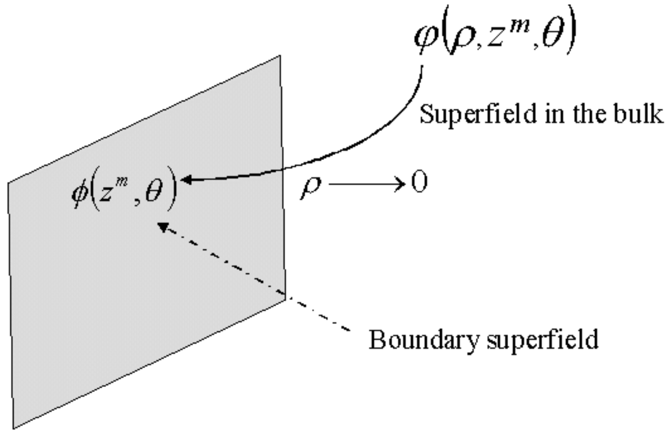
The strategy to construct the boundary superfields is the following. First we construct the supermultiplets on the bulk by acting on the abstract states spanning the UIR with the coset representative of the solvable superspace $AdS^{(Solv)}_{4|2\mathcal{N}}$ and then we reach the boundary by performing the limit $\rho \to 0$ (see fig. 2.4).

Then, we restrict us to the case $\mathcal{N}=2$. According to our previous discussion each of the $\mathcal{N}=2$ shortened multiplets, namely, the short multiplets, the hypermultiplet and the massless multiplets, correspond to a primary superfield on the boundary. We determine such superfields with the above described method. Short supermultiplets correspond to *constrained superfields*. The shortening conditions relating masses and hypercharges are retrieved here as the necessary condition to maintain the constraints after a superconformal transformation.

2.5.1 AdS_4 and ∂AdS_4 as cosets and their Killing vectors

We have previously studied $Osp(\mathcal{N}|4)$ and its representations in two different bases. The form (2.1.21) of the superalgebra is that used to construct the $Osp(2|4)$ and $Osp(3|4)$ supermultiplets in section 2.4. I showed, in section 12, how to translate these results in terms of the form (2.1.26) of the $Osp(\mathcal{N}|4)$ algebra in order to allow a comparison

Figure 2.4: Boundary superfields are obtained as limiting values of superfields on the bulk



with the three-dimensional CFT on the boundary. Now we introduce the description of the anti–de Sitter superspace and of its boundary in terms of supersolvable Lie algebra parametrization as in eq.s (2.5.5), (2.5.6). It turns out that such a description is the most appropriate for a comparative study between AdS_4 and its boundary. We calculate the Killing vectors of these two coset spaces since they are needed to determine the superfield multiplets living on both AdS_4 and ∂AdS_4.

So we write both the bulk and the boundary superspaces as supercosets[5],

$$\frac{G}{H}. \tag{2.5.9}$$

Applying supergroup elements $g \in Osp(\mathcal{N}|4)$ to the coset representatives $L(y)$ these latter transform as follows:

$$g\,L(y) = L(y')h(g, y)\,, \tag{2.5.10}$$

where $h(y)$ is some element of $H \subset Osp(\mathcal{N}|4)$, named the compensator that, generically depends both on g and on the coset point $y \in G/H$. For our purposes it is useful to consider the infinitesimal form of (2.5.10), i.e. for infinitesimal g we can write (see chapter 3):

$$\begin{aligned} g &= 1 + \epsilon^A T_A\,, \\ h &= 1 - \epsilon^A W_A^H(y) T_H\,, \\ y^{\mu\prime} &= y^\mu + \epsilon^A k_A^\mu(y) \end{aligned} \tag{2.5.11}$$

and we obtain:

$$T_A L(y) = k_A L(y) - L(y) T_H W_A^H(y)\,, \tag{2.5.12}$$

$$k_A \equiv k_A^\mu(y)\,\frac{\partial}{\partial y^\mu}\,. \tag{2.5.13}$$

[5]For an extensive explanation about supercosets I refer the reader to [37]. In the context of $D = 11$ and $D = 10$ compactifications see also [55].

The shifts in the superspace coordinates y determined by the supergroup elements (see eq.(2.5.10)) define the Killing vector fields (2.5.13) of the coset manifold.[6]

Let us now consider the solvable anti–de Sitter superspace defined in eq.s (2.5.5), (2.5.6). It describes a κ–gauge fixed supersymmetric extension of the bulk AdS_4. As explained by eq.(2.5.6) it is a supercoset (2.5.9) where $G = Osp(\mathcal{N}|4)$ and $H = CSO(1,2|\mathcal{N}) \times SO(\mathcal{N})$. Using the non–compact basis (2.1.26), the subgroup H is given by,

$$H^{AdS} = CSO(1,2|\mathcal{N}) \equiv \text{span} \left\{ J^m, K_m, s^i_\alpha, T^{ij} \right\} . \tag{2.5.14}$$

A coset representative can be written as follows[7]:

$$L^{AdS}(y) = \exp\left[\rho D + i\, x \cdot P + \theta^i q^i\right] , \qquad y = (\rho, x, \theta) . \tag{2.5.15}$$

In $AdS_{4|2\mathcal{N}}$ s-supersymmetry and K-symmetry have a non linear realization since the corresponding generators are not part of the solvable superalgebra $Ssolv_{adS}$ that is exponentiated (see eq.(2.5.8)).

The form of the Killing vectors simplifies considerably if we rewrite the coset representative as a product of exponentials

$$L(y) = \exp\left[i\, z \cdot P\right] \cdot \exp\left[\xi^i q^i\right] \cdot \exp\left[\rho D\right] . \tag{2.5.16}$$

This amounts to the following coordinate change:

$$\begin{aligned} z &= \left(1 - \tfrac{1}{2}\rho + \tfrac{1}{6}\rho^2 + \mathcal{O}(\rho^3)\right) x\, , \\ \xi^i &= \left(1 - \tfrac{1}{4}\rho + \tfrac{1}{24}\rho^2 + \mathcal{O}(\rho^3)\right) \theta^i\, . \end{aligned} \tag{2.5.17}$$

This is the parametrization that was used in [54] to get the $Osp(8|4)$-singleton action from the supermembrane. For this choice of coordinates the anti–de Sitter metric takes the standard form (2.5.4). The Killing vectors are

$$\begin{aligned} \vec{k}\,[P_m] &= -i\, \partial_m \, , \\ \vec{k}\,[q^{\alpha i}] &= \frac{\partial}{\partial \xi^i_\alpha} - \frac{1}{2} \left(\gamma^m \xi^i\right)^\alpha \partial_m \, , \\ \vec{k}\,[J^m] &= \varepsilon^{mpq} z_p \partial_q - \frac{i}{2} \left(\xi^i \gamma^m\right)_\alpha \frac{\partial}{\partial \xi^i_\alpha} \, , \\ \vec{k}\,[D] &= \frac{\partial}{\partial \rho} - z \cdot \partial - \frac{1}{2} \xi^i_\alpha \frac{\partial}{\partial \xi^i_\alpha} \, , \\ \vec{k}\,[s^{\alpha i}] &= -\xi^{\alpha i} \frac{\partial}{\partial \rho} + \frac{1}{2} \xi^{\alpha i}\, z \cdot \partial + \frac{i}{2} \varepsilon^{pqm}\, z_p (\gamma_q \xi^i)^\alpha \partial_m + \\ &\quad - \frac{1}{8} (\xi^j \xi^j)(\gamma^m \xi^i)^\alpha \partial_m - z^m (\gamma_m)^\alpha{}_\beta \frac{\partial}{\partial \xi^i_\beta} - \frac{1}{4} (\xi^j \xi^j) \frac{\partial}{\partial \xi^i_\alpha} + \\ &\quad + \frac{1}{2} \xi^{\alpha i} \xi^{\beta j} \frac{\partial}{\partial \xi^j_\beta} - \frac{1}{2} (\gamma^m \xi^i)^\alpha \xi^j_\beta \gamma_m \frac{\partial}{\partial \xi^j_\beta} \, , \end{aligned} \tag{2.5.18}$$

[6]The Killing vectors satisfy the algebra with structure functions with opposite sign, see [37].

[7]We use the notation $x \cdot y \equiv x^m y_m$ and $\theta^i q^i \equiv \theta^i_\alpha q^{\alpha i}$.

and for the compensators we find:

$$\begin{aligned}
W[P] &= 0\,, \\
W[q^{\alpha i}] &= 0\,, \\
W[J^m] &= -J^m\,, \\
W[D] &= 0\,, \\
W[s^{\alpha i}] &= -s^{\alpha i} + i\,\left(\gamma^m \theta^i\right)^\alpha\, J_m - i\theta^{\alpha j}\, T^{ij}\,.
\end{aligned} \tag{2.5.19}$$

For a detailed derivation of these Killing vectors and compensators I refer the reader to [15].

The boundary superspace $\partial(AdS_{4|2\mathcal{N}})$ is formed by the points on the supercoset with $\rho = 0$:

$$L^{CFT}(y) = \exp\left[\ \mathrm{i}\, x \cdot P + \theta^i q^i\right]\,. \tag{2.5.20}$$

In order to see how the supergroup acts on fields that live on this boundary we use the fact that this submanifold is by itself a supercoset. Indeed instead of $H^{AdS} \subset Osp(\mathcal{N}|4)$ as given in (2.5.14), we can choose the larger subalgebra

$$H^{CFT} = \mathrm{span}\left\{D, J^m, K_m, s^i_\alpha, T^{ij}\right\}\,, \tag{2.5.21}$$

and consider the new supercoset G/H^{CFT}. By definition also on this smaller space we have a non linear realization of the full orthosymplectic superalgebra. For the Killing vectors we find (see [15]):

$$\begin{aligned}
\vec{k}\,[P_m] &= -i\,\partial_m\,, \\
\vec{k}\,[q^{\alpha i}] &= \frac{\partial}{\partial\theta^i_\alpha} - \frac{1}{2}\left(\gamma^m \theta^i\right)^\alpha \partial_m\,, \\
\vec{k}\,[J^m] &= \varepsilon^{mpq} x_p \partial_q - \frac{i}{2}\left(\theta^i \gamma^m\right)_\alpha \frac{\partial}{\partial\theta^i_\alpha}\,, \\
\vec{k}\,[D] &= -x\cdot\partial - \frac{1}{2}\theta^i_\alpha \frac{\partial}{\partial\theta^i_\alpha}\,, \\
\vec{k}\,[s^{\alpha i}] &= \frac{1}{2}\theta^{\alpha i}\, x\cdot\partial + \frac{i}{2}\varepsilon^{pqm}\, x_p(\gamma_q\theta^i)^\alpha\partial_m - \frac{1}{8}(\theta^j\theta^j)(\gamma^m\theta^i)^\alpha\partial_m + \\
&\quad -x^m(\gamma_m)^\alpha{}_\beta \frac{\partial}{\partial\theta^i_\beta} - \frac{1}{4}(\theta^j\theta^j)\frac{\partial}{\partial\theta^i_\alpha} + \frac{1}{2}\theta^{\alpha i}\theta^{\beta j}\frac{\partial}{\partial\theta^j_\beta} - \frac{1}{2}(\gamma^m\theta^i)^\alpha\theta^j_\beta\gamma_m\frac{\partial}{\partial\theta^j_\beta}\,,
\end{aligned} \tag{2.5.22}$$

and for the compensators we have:

$$\begin{aligned}
W[P_m] &= 0\,, \\
W[q^{\alpha i}] &= 0\,, \\
W[J^m] &= -J^m\,, \\
W[D] &= D\,, \\
W[s^{\alpha i}] &= \theta^{\alpha i}\, D - s^{\alpha i} + i\,\left(\gamma^m \theta^i\right)^\alpha\, J_m - i\theta^j T^{ij}\,.
\end{aligned} \tag{2.5.23}$$

If we compare the Killing vectors on the boundary (2.5.22) with those on the bulk (2.5.18) we see that they are very similar. The only formal difference is the suppression of the $\frac{\partial}{\partial\rho}$

terms. The conceptual difference, however, is relevant. On the boundary the transformations generated by (2.5.22) are the *standard superconformal transformations* in three–dimensional (compactified) Minkowski space. On the bulk the transformations generated by (2.5.18) are *superisometries* of anti–de Sitter superspace. They might be written in completely different but equivalent forms if we used other coordinate frames. The form they have is due to the use of the *solvable coordinate frame* (ρ, z, ξ) which is the most appropriate to study the restriction of bulk supermultiplets to the boundary. For more details on this point I refer the reader to [15].

2.5.2 Conformal $Osp(2|4)$ superfields: general discussion

Let us restrict our attention to $\mathcal{N}$=2. In this case the $SO(2)$ group has just one generator that we name the hypercharge:

$$Y \equiv T^{12} \,. \tag{2.5.24}$$

Since it is convenient to work with eigenstates of the hypercharge operator, we reorganize the two Grassmann spinor coordinates of superspace in complex combinations:

$$\theta^{\pm}_{\alpha} = \frac{1}{\sqrt{2}}(\theta^1_\alpha \pm i\theta^2_\alpha) \,, \qquad Y\,\theta^{\pm}_{\alpha} = \pm\theta^{\pm}_{\alpha} \ . \tag{2.5.25}$$

In this new notations the Killing vectors generating q–supersymmetries on the boundary (see eq.(2.5.22)) take the form:

$$\vec{k}\left[q^{\alpha i}\right] \quad \longrightarrow \quad q^{\alpha\pm} = \frac{\partial}{\partial\theta^{\mp}_{\alpha}} - \tfrac{1}{2}\left(\gamma^m\right)^{\alpha}{}_{\beta}\theta^{\beta\pm}\partial_m \,. \tag{2.5.26}$$

A generic superfield is a function $\Phi(x,\theta)$ of the bosonic coordinates x and of all the $\theta.s$. Expanding such a field in power series of the $\theta.s$ we obtain a multiplet of x–space fields that, under the action of the Killing vector (2.5.26), form a representation of Poincaré supersymmetry. Such a representation can be shortened by imposing on the superfield $\Phi(x,\theta)$ constraints that are invariant with respect to the action of the Killing vectors (2.5.26). This is possible because of the existence of the so called superderivatives, namely of fermionic vector fields that commute with the supersymmetry Killing vectors. In our notations the superderivatives are defined as follows:

$$\mathcal{D}^{\alpha\pm} = \frac{\partial}{\partial\theta^{\mp}_{\alpha}} + \tfrac{1}{2}\left(\gamma^m\right)^{\alpha}{}_{\beta}\theta^{\beta\pm}\partial_m \,, \tag{2.5.27}$$

and satisfy the required property

$$\{\mathcal{D}^{\alpha\pm}, q^{\beta\pm}\} \;=\; \{\mathcal{D}^{\alpha\pm}, q^{\beta\mp}\} \;=\; 0 \,. \tag{2.5.28}$$

As explained in [37] the existence of superderivatives is the manifestation at the fermionic level of a general property of coset manifolds. For G/H the true isometry algebra is not G, rather it is $G \times (N(H)_G/H)$ (minus the explicit $U(1)$'s) where $N(H)_G$ denotes the normalizer of the stability subalgebra H (see chapter 3). The additional isometries are generated by *right–invariant* rather than *left–invariant* vector fields that as such commute with the *left–invariant* ones. If we agree that the Killing vectors are left–invariant vector fields then the superderivatives are right–invariant ones and generate the additional

superisometries of Poincaré superspace. Shortened representations of Poincaré supersymmetry are superfields with a prescribed behaviour under the additional superisometries: for instance they may be invariant under such transformations. We can formulate these shortening conditions by writing constraints such as

$$\mathcal{D}^{\alpha+}\Phi(x,\theta) = 0\,. \tag{2.5.29}$$

The key point in our discussion is that a constraint of type (2.5.29) is guaranteed from eq.s (2.5.28) to be invariant with respect to the superPoincaré algebra, yet it is not a priori guaranteed that it is invariant under the action of the full superconformal algebra (2.5.22). Investigating the additional conditions that make a constraint such as (2.5.29) superconformal invariant is the main goal of the present section. This is the main tool that allows a transcription of the Kaluza–Klein results for supermultiplets into a superconformal language.

To develop such a programme it is useful to perform a further coordinate change that is quite traditional in superspace literature [56]. Given the coordinates x on the boundary (or the coordinates z for the bulk) we set:

$$y^m = x^m + \tfrac{1}{2}\,\theta^+\gamma^m\theta^-\,. \tag{2.5.30}$$

Then the superderivatives become

$$\begin{aligned}\mathcal{D}^{\alpha+} &= \frac{\partial}{\partial\theta^-_\alpha}\,,\\ \mathcal{D}^{\alpha-} &= \frac{\partial}{\partial\theta^+_\alpha} + (\gamma^m)^\alpha{}_\beta\theta^{\beta-}\partial_m\,.\end{aligned} \tag{2.5.31}$$

It is our aim to describe superfield multiplets both on the bulk and on the boundary. It is clear that one can do the same redefinitions for the Killing vector of q-supersymmetry (2.5.26) and that one can introduce superderivatives also for the theory on the bulk.

So let us finally turn to superfields. We begin by focusing on boundary superfields since their treatment is slightly easier than the treatment of bulk superfields.

A **primary superfield** *is defined as follows (see [47], [57]):*

$$\Phi^{\partial AdS}(x,\theta) = \exp\left[i\,x\cdot P + \theta^i q^i\right]\Phi(0)\,, \tag{2.5.32}$$

where $\Phi(0)$ *is a primary field (see eq.(2.3.5))* [8]

$$\begin{aligned} s^i_\alpha\Phi(0) &= 0\,,\\ K_m\Phi(0) &= 0\,,\end{aligned} \tag{2.5.33}$$

of scaling weight D_0*, hypercharge* y_0 *and eigenvalue* j *for the "third-component" operator* J_2

$$D\,\Phi(0) = D_0\,\Phi(0) \quad;\quad Y\,\Phi(0) = y_0\,\Phi(0) \quad;\quad J_2\,\Phi(0) = j\,\Phi(0)\,. \tag{2.5.34}$$

From the above definition one sees that the primary superfield $\Phi^{\partial AdS}(x,\theta)$ is actually obtained by acting with the coset representative (2.5.20) on the $SO(1,2)\times SO(1,1)$-primary field. Hence we know how it transforms under the infinitesimal transformations

[8]For an operator Φ, the action of algebra operators is actually the adjoint action $[\mathcal{O},\Phi]$; however, as a shorthand notation we call the states with the names of the corresponding fields, so we write $\mathcal{O}\Phi$.

of the group $Osp(2|4)$. Indeed one simply uses (2.5.12) to obtain the result. For example under dilatation we have:

$$D\,\Phi^{\partial AdS}(x,\theta) = \left(-x\cdot\partial - \tfrac{1}{2}\theta^i\frac{\partial}{\partial\theta^i} + D_0\right)\Phi(x,\theta), \tag{2.5.35}$$

where the term D_0 comes from the compensator in (2.5.23). Of particular interest is the transformation under special supersymmetry since it imposes the constraints for shortening,

$$s^{\pm}\Phi^{\partial AdS}(x,\theta) = \vec{k}\,[s^{\pm}]\Phi(x,\theta) + e^{i\,x\cdot P+\theta^i q^i}\left(-\theta^{\pm}D - i\,\gamma^m\theta^{\pm}\,J_m + s^{\pm} \pm \theta^{\pm}Y\right)\Phi(0)\,. \tag{2.5.36}$$

For completeness we give the form of $s^{\pm}$ in the y-basis where it gets a relatively concise form,

$$\begin{aligned}\vec{k}\,[s^{\alpha -}] &= -\left(y\cdot\gamma\right)^{\alpha}{}_{\beta}\frac{\partial}{\partial\theta^{+}_{\beta}} + \frac{1}{2}\left(\theta^{-}\theta^{-}\right)\frac{\partial}{\partial\theta^{-}_{\alpha}}\\ \vec{k}\,[s^{\alpha +}] &= \theta^{\alpha +}y\cdot\partial + i\,\varepsilon^{pqm}y_p\left(\gamma_p\theta^{+}\right)^{\alpha}\partial_m + \frac{1}{2}\left(\theta^{+}\theta^{+}\right)\frac{\partial}{\partial\theta^{+}_{\alpha}} +\\ &\quad +\theta^{+}\gamma^m\theta^{-}\left(\gamma_m\right)^{\alpha}{}_{\beta}\frac{\partial}{\partial\theta^{-}_{\beta}}\,.\end{aligned} \tag{2.5.37}$$

Let us now turn to a direct discussion of multiplet shortening and consider the superconformal invariance of Poincaré constraints constructed with the superderivatives $\mathcal{D}^{\alpha\pm}$. The simplest example is provided by the *chiral supermultiplet*. By definition this is a scalar superfield $\Phi_{chiral}(y,\theta)$ obeying the constraint (2.5.29) which is solved by boosting only along q^- and not along q^+:

$$\Phi_{chiral}(y,\theta) = e^{i\,y\cdot P+\theta^{+}q^{-}}\,\Phi(0). \tag{2.5.38}$$

Hence we have

$$\Phi_{chiral}(\rho,y,\theta) = X(\rho,y) + \theta^{+}\lambda(\rho,y) + \theta^{+}\theta^{+}H(\rho,y) \tag{2.5.39}$$

on the bulk or

$$\Phi_{chiral}(y,\theta) = X(y) + \theta^{+}\lambda(y) + \theta^{+}\theta^{+}H(y) \tag{2.5.40}$$

on the boundary. The field components of the chiral multiplet are:

$$X = e^{i\,y\cdot P}\,\Phi(0)\,,\qquad \lambda = i\,e^{i\,y\cdot P}\,q^{-}\Phi(0)\,,\qquad H = -\tfrac{1}{4}e^{i\,y\cdot P}\,q^{-}q^{-}\phi(0)\,. \tag{2.5.41}$$

For completeness, we write the superfield Φ also in the x-basis[9],

$$\begin{aligned}\Phi(x) &= X(x) + \theta^{+}\lambda(x) + (\theta^{+}\theta^{+})H(x) + \tfrac{1}{2}\theta^{+}\gamma^m\theta^{-}\partial_m X(x) +\\ &\quad +\tfrac{1}{4}(\theta^{+}\theta^{+})\theta^{-}\partial\!\!\!/\lambda(x) + \tfrac{1}{16}(\theta^{+}\theta^{+})(\theta^{-}\theta^{-})\Box X(x) =\\ &= \exp\left(\tfrac{1}{2}\theta^{+}\gamma^m\theta^{-}\partial_m\right)\Phi(y)\,.\end{aligned} \tag{2.5.42}$$

[9] where $\Box = \partial^m\partial_m$.

Because of (2.5.28), we are guaranteed that under q–supersymmetry the chiral superfield Φ_{chiral} transforms into a chiral superfield. We should verify that this is true also for s–supersymmetry. To say it simply we just have to check that $s^-\Phi_{chiral}$ does not depend on θ^-. This is not generically true, but it becomes true if certain extra conditions on the quantum numbers of the primary state are satisfied. Such conditions are the same one obtains as multiplet shortening conditions when constructing the UIRs of the superalgebra.

In the specific instance of the chiral multiplet, looking at (2.5.36) and (2.5.37) we see that in $s^-\Phi_{chiral}$ the terms depending on θ^- are the following ones:

$$s^-\Phi\Big|_{\theta^-} = -\left(D_0 + y_0\right)\theta^-\Phi = 0\,, \tag{2.5.43}$$

they cancel if

$$D_0 = -y_0\,. \tag{2.5.44}$$

Eq.(2.5.44) is easily recognized as the unitarity condition for the existence of $Osp(2|4)$ hypermultiplets (see section 2.4). The algebra (2.1.26) ensures that the chiral multiplet also transforms into a chiral multiplet under K_m. Moreover we know that the action of the compensators of K_m on the chiral multiplet is zero. Furthermore, the compensators of the generators P_m, q^i, J_m on the chiral multiplet are zero and from (2.5.12) we conclude that their generators act on the chiral multiplet as the Killing vectors.

Notice that the linear part of the s-supersymmetry transformation on the chiral multiplet has the same form of the q-supersymmetry but with the parameter taken to be $\epsilon_q = -i\,y\cdot\gamma\epsilon_s$. As already stated the non-linear form of s-supersymmetry is the consequence of its gauge fixing which we have implicitly imposed from the start by choosing the supersolvable Lie algebra parametrization of superspace and by taking the coset representatives as in (2.5.15) and (2.5.20). In addition to the chiral multiplet there exists also the complex conjugate *antichiral multiplet* $\bar\Phi_{chiral} = \Phi_{antichiral}$ with opposite hypercharge and the relation $D_0 = y_0$.

2.5.3 Matching the Kaluza Klein results for $Osp(2|4)$ supermultiplets with boundary conformal superfields

It is now our purpose to reformulate the $\mathcal{N}=2$ multiplets in terms of superfields living on the boundary of the AdS_4 space–time manifold. This is the key step to convert information coming from classical harmonic analysis on the compact manifold X_7 into predictions on the spectrum of conformal primary operators present in the three–dimensional gauge theory of the M2–brane.

Interpreted as superfields on the boundary the *long multiplets* correspond to *unconstrained superfields* and their discussion is quite straightforward. We are mostly interested in short multiplets that correspond to composite operators of the microscopic gauge theory with protected scaling dimensions. In superfield language, as we have shown in the previous section, *short multiplets* are constrained superfields.

Just as on the boundary, also on the bulk, we obtain such constraints by means of the bulk superderivatives. In order to show how this works we begin by discussing the *chiral superfield on the bulk* and then show how it is obtained from the hypermultiplet.

Chiral superfields are the Hypermultiplets: the basic example

The treatment for the bulk chiral field is completely analogous to that of chiral superfield on the boundary.

Generically bulk superfields are given by:

$$\Phi^{AdS}(\rho, x, \theta) = \exp\left[\rho D + \mathrm{i}\, x \cdot P + \theta^i q^i\right] \, \Phi(0)\,. \tag{2.5.45}$$

Using the parametrization (2.5.17) we can rewrite (2.5.45) in the following way:

$$\Phi^{AdS}(\rho, z, \xi) = \exp\left[\mathrm{i}\, z \cdot P + \xi^i q^i\right] \, \cdot \, \exp\left[\rho D_0\right] \, \Phi(0)\,. \tag{2.5.46}$$

Then the generator D acts on this field as follows:

$$D\, \Phi^{AdS}(\rho, z, \xi) = \left(-z \cdot \partial - \tfrac{1}{2}\xi^i \frac{\partial}{\partial \xi^i} + D_0\right) \Phi^{AdS}(\rho, z, \xi)\,. \tag{2.5.47}$$

Just as for boundary chiral superfields, also on the bulk we find that the constraint (2.5.29) is invariant under the s-supersymmetry rule (2.5.18) if and only if:

$$D_0 = -y_0\,. \tag{2.5.48}$$

Furthermore, looking at (2.5.46) one sees that for the bulk superfields $D_0 = 0$ is forbidden. This constraint on the scaling dimension together with the relation $E_0 = -D_0$, coincides with the constraint:

$$E_0 = |y_0| \tag{2.5.49}$$

defining the $Osp(2|4)$ hypermultiplet UIR of $Osp(2|4)$. The transformation of the bulk chiral superfield under s, P_m, q^i, J_m is simply given by the bulk Killing vectors. In particular the form of the s-supersymmetry Killing vector coincides with that given in (2.5.37) for the boundary.

As we saw a chiral superfield on the bulk describes an $Osp(2|4)$ hypermultiplet.

Applying the rotation matrix U of eq. (2.3.8) to the states in table 2.7 we indeed find the field components (2.5.41) of the chiral supermultiplet [10].

Having clarified how to obtain the four-dimensional chiral superfield from the $Osp(2|4)$ hypermultiplet we can now obtain the other shortened $Osp(2|4)$ superfields from the supermultiplets found in section 2.4.4.

Superfield description of the short vector multiplet

Let us start with the short massive vector multiplet. The constraint for shortening is

$$E_0 = |y_0| + 1 \tag{2.5.50}$$

and the particle states of the multiplet are given in table 2.6. Applying the rotation matrix U to the states in table 2.6 we find the following states:

$$S = |\mathrm{vac}\rangle\,, \qquad \lambda_L^{\pm} = i\, q^{\pm}|\mathrm{vac}\rangle\,, \qquad \pi^{--} = -\tfrac{1}{4}\, q^- q^- |\mathrm{vac}\rangle\,, \qquad \mathrm{etc}\ldots \tag{2.5.51}$$

[10] I remind that the fields on the bulk are on–shell, the fields on the boundary are off–shell; then, for example, the spinor in table 2.7 has the same number of degrees of freedom of the spinor in (2.5.40), that is two, because the first is four dimensional on–shell, the second is three dimensional off–shell.

where we used the same notation for the rotated as for the original states and up to an irrelevant factor $\frac{1}{4}$. We follow the same procedure also for the other short and massless multiplets. Namely in the superfield transcription of our multiplets we use the same names for the superspace field components as for the particle fields appearing in the $SO(3)\times SO(2)$ basis. Moreover when convenient we rescale some field components without mentioning it explicitly. The list of states appearing in (2.5.51) are the components of a superfield

$$\Phi_{vector} \;=\; S+\theta^-\lambda_L^+ +\theta^+\lambda_L^- +\theta^+\theta^-\pi^0+\theta^+\theta^+\pi^{--}+\theta^+\not{A}\theta^- +\theta^+\theta^+\,\theta^-\lambda_T^-\,, \tag{2.5.52}$$

which is the explicit solution of the following constraint

$$\mathcal{D}^+\mathcal{D}^+\Phi_{vector}=0\,. \tag{2.5.53}$$

imposed on a superfield of the form (2.5.45) with hypercharge y_0.

In superspace literature a superfield of type (2.5.52) is named a linear superfield. If we consider the variation of a linear superfield with respect to s^-, such variation contains, a priori, a term of the form

$$s^-\Phi_{vector}\Big|_{\theta^-\theta^-}=\tfrac{1}{2}\,(D_0+y_0+1)\,(\theta^-\theta^-)\lambda_L^+\,, \tag{2.5.54}$$

which has to cancel if Φ_{vector} is to transform into a linear multiplet under s^-. Hence the following condition has to be imposed

$$D_0=-y_0-1\,. \tag{2.5.55}$$

which is identical with the bound for the vector multiplet shortening $E_0=y_0+1$.

Superfield description of the short gravitino multiplet

Let us consider the short gravitino multiplets. The particle state content of these multiplets is given in table 2.5. Applying the rotation matrix U (2.3.8) to these states, and identifying the particle states with the corresponding rotated field states as we have done in the previous cases, we find the following spinorial superfield

$$\begin{aligned}\Phi_{gravitino} \;=\;& \lambda_L+\not{A}^+\theta^- +\not{A}^-\theta^+ +\not{\phi}^-\theta^+ +3\,(\theta^+\theta^-)\lambda_T^{+-}-(\theta^+\gamma^m\theta^-)\gamma_m\lambda_T^{+-}+\\ &+(\theta^+\theta^+)\lambda_T^{--}+(\theta^+\gamma^m\theta^-)\chi_m^{(+)}+(\theta^+\theta^+)\not{Z}^-\theta^-\,,\end{aligned} \tag{2.5.56}$$

where the vector–spinor field χ^m is expressed in terms of the spin-$\frac{3}{2}$ field with symmetrized spinor indices in the following way

$$\chi^{(+)m\alpha}=(\gamma^m)_{\beta\gamma}\,\chi^{(+)(\alpha\beta\gamma)} \tag{2.5.57}$$

and where, as usual, $\not{A}^+=\gamma^m\,A_m^+$.

The superfield $\Phi_{gravitino}$ is linear in the sense that it does not depend on the monomial $\theta^-\theta^-$, but to be precise it is a spinorial superfield (2.5.45) with hypercharge y_0 that fulfills the stronger constraint

$$\mathcal{D}_\alpha^+\Phi^\alpha_{gravitino}=0\,. \tag{2.5.58}$$

The generic linear spinor superfield contains, in its expansion, also terms of the form $\varphi^+\theta^-$ and $(\theta^+\theta^+)\varphi^-\theta^-$, where φ^+ and φ^- are scalar fields and a term $(\theta^+\gamma^m\theta^-)\chi_m$ where the spinor-vector χ_m is not an irreducible $\frac{3}{2}$ representation since it cannot be written as in (2.5.57).

Explicitly we have:

$$\begin{aligned}\Phi^\alpha_{linear} &= \lambda_L + \not{A}^+\theta^- + \not{A}^-\theta^+ + \not{\phi}^-\theta^+ + \varphi^+\theta^- + 3\,(\theta^+\theta^-)\lambda_T^{+-} + (\theta^+\theta^+)\lambda_T^{--} + \\ &\quad +(\theta^+\gamma^m\theta^-)\chi_m + (\theta^+\theta^+)\not{Z}^-\theta^- + (\theta^+\theta^+)\varphi^-\theta^-\,. \end{aligned} \tag{2.5.59}$$

The field component $\chi^{\alpha m}$ in a generic unconstrained spinor superfield can be decomposed in a spin-$\frac{1}{2}$ component and a spin-$\frac{3}{2}$ component according to,

$$\ydiagram{2} \times \ydiagram{1} = \ydiagram{2,1} + \ydiagram{3} \tag{2.5.60}$$

where $m = \ydiagram{2}$. Then the constraint (2.5.58) eliminates the scalars $\varphi^\pm$ and eliminates the $\ydiagram{2,1}$ -component of χ in terms of λ_T^{+-}. From

$$s^-_\beta\,\Phi^\alpha_{gravitino}\Big|_{\theta^-\theta^-} = \tfrac{1}{2}\left(-D_0 - y_0 - \tfrac{3}{2}\right)(\theta^-\theta^-)(\not{A}^+)_\beta{}^\alpha \tag{2.5.61}$$

we conclude that the constraint (2.5.58) is superconformal invariant if and only if

$$D_0 = -y_0 - \tfrac{3}{2}\,. \tag{2.5.62}$$

Once again we have retrieved the shortening condition already known in the $SO(3)\times SO(2)$ basis: $E_0 = |y_0| + \frac{3}{2}$.

Superfield description of the short graviton multiplet

Applying the rotation U (2.3.8) to the states of table 2.4, and identifying the particle states with the corresponding boundary fields, as we have done so far, we derive the short graviton superfield:

$$\begin{aligned}\Phi^m_{graviton} &= A^m + \theta^+\gamma^m\lambda_T^- + \theta^-\chi^{(+)+m} + \theta^+\chi^{(+)-m} + \\ &\quad +(\theta^+\theta^-)\,Z^{+-m} + \tfrac{i}{2}\,\varepsilon^{mnp}\,(\theta^+\gamma_n\theta^-)\,Z_p^{+-} + +(\theta^+\theta^+)\,Z^{--m} \\ &\quad +(\theta^+\gamma_n\theta^-)\,h^{mn} + (\theta^+\theta^+)\,\theta^-\chi^{(-)-m}\,, \end{aligned} \tag{2.5.63}$$

where

$$\begin{aligned}\chi^{(+)\pm m\alpha} &= (\gamma^m)_{\beta\gamma}\,\chi^{(+)\pm(\alpha\beta\gamma)}\,, \\ \chi^{(-)-m\alpha} &= (\gamma^m)_{\beta\gamma}\,\chi^{(-)-(\alpha\beta\gamma)}\,, \\ h^m{}_m &= 0\,. \end{aligned} \tag{2.5.64}$$

This superfield satisfies the following constraint,

$$\mathcal{D}^+_\alpha\Phi^{\alpha\beta}_{graviton} = 0\,, \tag{2.5.65}$$

where we have defined:

$$\Phi^{\alpha\beta} = (\gamma_m)^{\alpha\beta}\ \Phi^m\,. \tag{2.5.66}$$

Furthermore we check that $s^-\Phi^m_{graviton}$ is still a short graviton superfield if and only if:

$$D_0 = -y_0 - 2\,. \tag{2.5.67}$$

corresponding to the known unitarity bound:

$$E_0 = |y_0| + 2\,. \tag{2.5.68}$$

Superfield description of the massless vector multiplet

Considering now massless multiplets we focus on the massless vector multiplet, described in table 2.10. Applying the rotation U (2.3.8) we get,

$$V = S + \theta^+\lambda_L^- + \theta^-\lambda_L^+ + (\theta^+\theta^-)\,\pi + \theta^+ \not{A}\theta^-\,. \tag{2.5.69}$$

This multiplet can be obtained by a real superfield

$$\begin{aligned} V &= S + \theta^+\lambda_L^- + \theta^-\lambda_L^+ + (\theta^+\theta^-)\,\pi + \theta^+\not{A}\theta^- + \\ &\quad +(\theta^+\theta^+)\,M^{--} + (\theta^-\theta^-)\,M^{++} + \\ &\quad +(\theta^+\theta^+)\,\theta^-\mu^- + (\theta^-\theta^-)\,\theta^+\mu^+ + \\ &\quad +(\theta^+\theta^+)(\theta^-\theta^-)\,F\,, \\ V^\dagger &= V \end{aligned} \tag{2.5.70}$$

that transforms as follows under a gauge transformation,

$$V \rightarrow V + \Lambda + \Lambda^\dagger\,, \tag{2.5.71}$$

where Λ is a chiral superfield of the form (2.5.42). In components this reads,

$$\begin{aligned} S &\longrightarrow S + X + X^*\,, \\ \lambda_L^- &\longrightarrow \lambda_L^- + \lambda\,, \\ \pi &\longrightarrow \pi\,, \\ A_m &\longrightarrow A_m + \tfrac{1}{2}\,\partial_m\,(X - X^*)\,, \\ M^{--} &\longrightarrow M^{--} + H\,, \\ \mu^- &\longrightarrow \mu^- + \tfrac{1}{4}\,\not{\partial}\lambda\,, \\ F &\longrightarrow F + \tfrac{1}{16}\,\Box X\,, \end{aligned} \tag{2.5.72}$$

which may be used to gauge fix the real multiplet in the following way,

$$M^{--} = M^{++} = \mu^- = \mu^+ = F = 0\,, \tag{2.5.73}$$

to obtain (2.5.69). For the scaling weight D_0 of the massless vector multiplet we find -1. Indeed this follows from the fact that Λ is a chiral superfield with $y_0 = 0, D_0 = 0$. Which is also in agreement with $E_0 = 1$.

Superfield description of the massless graviton multiplet

The massless graviton multiplet is composed of the bulk particle states listed in table 2.8, from which, with the usual procedure we obtain

$$g_m = A_m + \theta^+ \chi_m^{(+)-} + \theta^- \chi_m^{(+)+} + \theta^+ \gamma^n \theta^- \, h_{mn} \, . \tag{2.5.74}$$

Similarly as for the vector multiplet we may write this multiplet as a gauge fixed multiplet with local gauge symmetries that include local coordinate transformations, local supersymmetry and local $SO(2)$, in other words full supergravity. However this is not the goal of this chapter where we prepare to interprete the bulk gauge fields as composite states in the boundary conformal field theory.

This completes the treatment of the short $Osp(2|4)$ boundary superfields. We have found that all of them are linear superfields with the extra constraint that they have to transform into superfields of the same type under s-supersymmetry. Such constraint is identical to the shortening conditions found by representation theory of $Osp\,(2|4)$.

$SD(E_0 > y_0 + 2, 1, y_0 \geq 0\|2)$		
spin	energy	hypercharge
2	$E_0 + 1$	y_0
3/2	$E_0 + 3/2$	$y_0 - 1$
3/2	$E_0 + 3/2$	$y_0 + 1$
3/2	$E_0 + 1/2$	$y_0 - 1$
3/2	$E_0 + 1/2$	$y_0 + 1$
1	$E_0 + 2$	y_0
1	$E_0 + 1$	$y_0 - 2$
1	$E_0 + 1$	$y_0 + 2$
1	$E_0 + 1$	y_0
1	$E_0 + 1$	y_0
1	E_0	y_0
1/2	$E_0 + 3/2$	$y_0 - 1$
1/2	$E_0 + 3/2$	$y_0 + 1$
1/2	$E_0 + 1/2$	$y_0 - 1$
1/2	$E_0 + 1/2$	$y_0 + 1$
0	$E_0 + 1$	y_0

Table 2.1: $\mathcal{N} = 2$ long graviton multiplet with $y_0 \geq 0$

$SD(E_0 > y_0 + 3/2, 1/2, y_0 \geq 0 \vert 2)$		
spin	energy	hypercharge
$3/2$	$E_0 + 1$	y_0
1	$E_0 + 3/2$	$y_0 - 1$
1	$E_0 + 3/2$	$y_0 + 1$
1	$E_0 + 1/2$	$y_0 - 1$
1	$E_0 + 1/2$	$y_0 + 1$
$1/2$	$E_0 + 2$	y_0
$1/2$	$E_0 + 1$	$y_0 - 2$
$1/2$	$E_0 + 1$	y_0
$1/2$	$E_0 + 1$	$y_0 + 2$
$1/2$	$E_0 + 1$	y_0
$1/2$	E_0	y_0
0	$E_0 + 3/2$	$y_0 - 1$
0	$E_0 + 3/2$	$y_0 + 1$
0	$E_0 + 1/2$	$y_0 - 1$
0	$E_0 + 1/2$	$y_0 + 1$

Table 2.2: $\mathcal{N} = 2$ long gravitino multiplet with $y_0 \geq 0$

$SD(E_0 > y_0 + 3/2, 0, y_0 \geq 0 \vert 2)$		
spin	energy	hypercharge
1	$E_0 + 1$	y_0
$1/2$	$E_0 + 3/2$	$y_0 - 1$
$1/2$	$E_0 + 3/2$	$y_0 + 1$
$1/2$	$E_0 + 1/2$	$y_0 - 1$
$1/2$	$E_0 + 1/2$	$y_0 + 1$
0	$E_0 + 2$	y_0
0	$E_0 + 1$	$y_0 - 2$
0	$E_0 + 1$	$y_0 + 2$
0	$E_0 + 1$	y_0
0	E_0	y_0

Table 2.3: $\mathcal{N} = 2$ long vector multiplet with $y_0 \geq 0$

$SD(y_0+2,1,y_0>0\|2)$		
spin	energy	hypercharge
2	y_0+3	y_0
3/2 3/2 3/2	$y_0+7/2$ $y_0+5/2$ $y_0+5/2$	y_0-1 y_0+1 y_0-1
1 1 1	y_0+3 y_0+3 y_0+2	y_0-2 y_0 y_0
1/2	$y_0+5/2$	y_0-1

Table 2.4: $\mathcal{N}=2$ short graviton multiplet with $y_0>0$

$SD(y_0+3/2,1/2,y_0>0\|2)$		
spin	energy	hypercharge
3/2	$y_0+5/2$	y_0
1 1 1	y_0+3 y_0+2 y_0+2	y_0-1 y_0+1 y_0-1
1/2 1/2 1/2	$y_0+5/2$ $y_0+5/2$ $y_0+3/2$	y_0 y_0-2 y_0
0	y_0+3	$y_0\pm1$

Table 2.5: $\mathcal{N}=2$ short gravitino multiplet with $y_0>0$

$SD(y_0+1,0,y_0>0\|2)$		
spin	energy	hypercharge
1	y_0+2	y_0
1/2 1/2 1/2	$y_0+5/2$ $y_0+3/2$ $y_0+3/2$	y_0-1 y_0+1 y_0-1
0 0 0	y_0+2 y_0+2 y_0+1	y_0-2 y_0 y_0

Table 2.6: $\mathcal{N}=2$ short vector multiplet with $y_0>0$

$SD(y_0,0,y_0>1/2\|2)$		
spin	energy	hypercharge
1/2	$y_0+1/2$	y_0-1
0 0	y_0+1 y_0	y_0-2 y_0

Table 2.7: $\mathcal{N}=2$ hypermultiplet with $y_0>1/2$

$SD(2,1,0\|2)$		
spin	energy	hypercharge
2	3	0
3/2 3/2	5/2 5/2	-1 $+1$
1	2	0

Table 2.8: $\mathcal{N}=2$ massless graviton multiplet

$SD(3/2,1/2,0\|2)$		
spin	energy	hypercharge
3/2	5/2	0
1 1	2 2	−1 +1
1/2	3/2	0

Table 2.9: $\mathcal{N}=2$ massless gravitino multiplet

$SD(1,0,0\|2)$		
spin	energy	hypercharge
1	2	0
1/2 1/2	3/2 3/2	−1 +1
0 0	2 1	0 0

Table 2.10: $\mathcal{N}=2$ massless vector multiplet

$SD(1/2,0,1/2\|2)$		
spin	energy	hypercharge
1/2	1	−1/2
0	1/2	1/2

Table 2.11: $\mathcal{N}=2$ supersingleton representation

$SD(E_0 > J_0 + 3/2, 1, J_0 \geq 2\|3)$		
spin	energy	isospin
2	$E_0 + \frac{3}{2}$	J_0
$\frac{3}{2}$	$E_0 + 2$	$J_0 + 1$ J_0 $J_0 - 1$
	$E_0 + 1$	$J_0 + 1$ J_0 $J_0 - 1$
1	$E_0 + \frac{5}{2}$	$J_0 + 1$ J_0 $J_0 - 1$
	$E_0 + \frac{3}{2}$	$J_0 + 2$ $J_0 + 1$ $J_0 + 1$ J_0 J_0 J_0 $J_0 - 1$ $J_0 - 1$ $J_0 - 2$
	$E_0 + \frac{1}{2}$	$J_0 + 1$ J_0 $J_0 - 1$
$\frac{1}{2}$	$E_0 + 3$	J_0
	$E_0 + 2$	$J_0 + 2$ $J_0 + 1$ $J_0 + 1$ J_0 J_0 J_0 $J_0 - 1$ $J_0 - 1$ $J_0 - 2$
	$E_0 + 1$	$J_0 + 2$ $J_0 + 1$ $J_0 + 1$ J_0 J_0 J_0 $J_0 - 1$ $J_0 - 1$ $J_0 - 2$
	E_0	J_0
0	$E_0 + \frac{5}{2}$	$J_0 + 1$ J_0 $J_0 - 1$
	$E_0 + \frac{3}{2}$	$J_0 + 2$ $J_0 + 1$ $J_0 + 1$ J_0 J_0 $J_0 - 1$ $J_0 - 1$ $J_0 - 2$
	$E_0 + \frac{1}{2}$	$J_0 + 1$ J_0 $J_0 - 1$

$SD(E_0 > 5/2, 1, 1\|3)$		
spin	energy	isospin
2	$E_0 + \frac{3}{2}$	1
$\frac{3}{2}$	$E_0 + 2$	2 1 0
	$E_0 + 1$	2 1 0
1	$E_0 + \frac{5}{2}$	2 1 0
	$E_0 + \frac{3}{2}$	3 2 2 1 1 1 0
	$E_0 + \frac{1}{2}$	2 1 0
$\frac{1}{2}$	$E_0 + 3$	1
	$E_0 + 2$	3 2 2 1 1 1 0
	$E_0 + 1$	3 2 2 1 1 1 0
	E_0	1
0	$E_0 + \frac{5}{2}$	2 1 0
	$E_0 + \frac{3}{2}$	3 2 2 1 1 0
	$E_0 + \frac{1}{2}$	2 1 0

$SD(E_0 > 3/2, 1, 0\|3)$		
spin	energy	isospin
2	$E_0 + \frac{3}{2}$	0
$\frac{3}{2}$	$E_0 + 2$	1
	$E_0 + 1$	1
1	$E_0 + \frac{5}{2}$	1
	$E_0 + \frac{3}{2}$	2 1 0
	$E_0 + \frac{1}{2}$	1
$\frac{1}{2}$	$E_0 + 3$	0
	$E_0 + 2$	2 1 0
	$E_0 + 1$	2 1 0
	E_0	0
0	$E_0 + \frac{5}{2}$	1
	$E_0 + \frac{3}{2}$	2 1
	$E_0 + \frac{1}{2}$	1

Table 2.12: The long $\mathcal{N} = 3$ graviton multiplet: $E_0 > J_0 + \frac{3}{2}$. From left to right: $J_0 \geq 2, J_0 = 1, J_0 = 0$.

$SD(E_0 > J_0+1, 1/2, J_0 \geq 2\|3)$		
spin	energy	isospin
$\frac{3}{2}$	$E_0+\frac{3}{2}$	J_0
1	E_0+2	$\begin{cases} J_0+1 \\ J_0 \\ J_0-1 \end{cases}$
	E_0+1	$\begin{cases} J_0+1 \\ J_0 \\ J_0-1 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{5}{2}$	$\begin{cases} J_0+1 \\ J_0 \\ J_0-1 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{3}{2}$	$\begin{cases} J_0+2 \\ J_0+1 \\ J_0+1 \\ J_0 \\ J_0 \\ J_0-1 \\ J_0-1 \\ J_0-2 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{1}{2}$	$\begin{cases} J_0+1 \\ J_0 \\ J_0-1 \end{cases}$
0	E_0+3	$\{ J_0$
	E_0+2	$\begin{cases} J_0+2 \\ J_0+1 \\ J_0 \\ J_0 \\ J_0-1 \\ J_0-2 \end{cases}$
	E_0+1	$\begin{cases} J_0+2 \\ J_0+1 \\ J_0 \\ J_0 \\ J_0-1 \\ J_0-2 \end{cases}$
	E_0	$\{ J_0$

$SD(E_0 > 2, 0, 1\|3)$		
spin	energy	isospin
$\frac{3}{2}$	$E_0+\frac{3}{2}$	1
1	E_0+2	$\begin{cases} 2 \\ 1 \\ 0 \end{cases}$
	E_0+1	$\begin{cases} 2 \\ 1 \\ 0 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{5}{2}$	$\begin{cases} 2 \\ 1 \\ 0 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{3}{2}$	$\begin{cases} 3 \\ 2 \\ 2 \\ 1 \\ 1 \\ 0 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{1}{2}$	$\begin{cases} 2 \\ 1 \\ 0 \end{cases}$
0	E_0+3	$\{ 1$
	E_0+2	$\begin{cases} 3 \\ 2 \\ 1 \\ 1 \end{cases}$
	E_0+1	$\begin{cases} 3 \\ 2 \\ 1 \\ 1 \end{cases}$
	E_0	$\{ 1$

$SD(E_0 > 1, 0, 0\|3)$		
spin	energy	isospin
$\frac{3}{2}$	$E_0+\frac{3}{2}$	0
1	E_0+2	$\{ 1$
	E_0+1	$\{ 1$
$\frac{1}{2}$	$E_0+\frac{5}{2}$	$\{ 1$
$\frac{1}{2}$	$E_0+\frac{3}{2}$	$\begin{cases} 2 \\ 1 \end{cases}$
$\frac{1}{2}$	$E_0+\frac{1}{2}$	$\{ 1$
0	E_0+3	$\{ 0$
	E_0+2	$\begin{cases} 2 \\ 0 \end{cases}$
	E_0+1	$\begin{cases} 2 \\ 0 \end{cases}$
	E_0	$\{ 0$

Table 2.13: The long $\mathcal{N}=3$ gravitino multiplet: $E_0 > J_0+1$. From left to right: $J_0 \geq 2, J_0 = 1, J_0 = 0$.

$SD(J_0+3/2, 1/2, J_0 \geq 2\vert 3)$		
spin	energy	isospin
2	J_0+3	J_0
$\frac{3}{2}$	$J_0+\frac{7}{2}$	$\left\{\begin{array}{l} J_0 \\ J_0-1 \end{array}\right.$
	$J_0+\frac{5}{2}$	$\left\{\begin{array}{l} J_0+1 \\ J_0 \\ J_0-1 \end{array}\right.$
1	J_0+4	$\{\, J_0-1$
	J_0+3	$\left\{\begin{array}{l} J_0+1 \\ J_0 \\ J_0 \\ J_0-1 \\ J_0-1 \\ J_0-2 \end{array}\right.$
	J_0+2	$\left\{\begin{array}{l} J_0+1 \\ J_0 \\ J_0-1 \end{array}\right.$
$\frac{1}{2}$	$J_0+\frac{7}{2}$	$\left\{\begin{array}{l} J_0 \\ J_0-1 \\ J_0-2 \end{array}\right.$
	$J_0+\frac{5}{2}$	$\left\{\begin{array}{l} J_0+1 \\ J_0 \\ J_0 \\ J_0-1 \\ J_0-1 \\ J_0-2 \end{array}\right.$
	$J_0+\frac{3}{2}$	$\{\, J_0$
0	J_0+3	$\left\{\begin{array}{l} J_0 \\ J_0-1 \\ J_0-2 \end{array}\right.$
	J_0+2	$\left\{\begin{array}{l} J_0 \\ J_0-1 \end{array}\right.$

$SD(5/2, 1/2, 1\vert 3)$		
spin	energy	isospin
2	4	1
$\frac{3}{2}$	$\frac{9}{2}$	$\left\{\begin{array}{l} 1 \\ 0 \end{array}\right.$
	$\frac{7}{2}$	$\left\{\begin{array}{l} 2 \\ 1 \\ 0 \end{array}\right.$
1	5	$\{\, 0$
	4	$\left\{\begin{array}{l} 2 \\ 1 \\ 1 \\ 0 \end{array}\right.$
	3	$\left\{\begin{array}{l} 2 \\ 1 \\ 0 \end{array}\right.$
$\frac{1}{2}$	$\frac{9}{2}$	$\{\, 1$
	$\frac{7}{2}$	$\left\{\begin{array}{l} 2 \\ 1 \\ 1 \\ 0 \end{array}\right.$
	$\frac{5}{2}$	$\{\, 1$
0	4	$\{\, 1$
	3	$\left\{\begin{array}{l} 1 \\ 0 \end{array}\right.$

$SD(3/2, 1/2, 0\vert 3)$		
spin	energy	isospin
2	3	0
$\frac{3}{2}$	$\frac{5}{2}$	$\{\, 1$
1	2	$\{\, 1$
$\frac{1}{2}$	$\frac{3}{2}$	$\{\, 0$

Table 2.14: The short $\mathcal{N}=3$ graviton multiplet: $E_0 = J_0 + \frac{3}{2}$. From left to right: $J_0 \geq 2$, $J_0 = 1$, and $J_0 = 0$ (that is massless).

$SD(J_0+1, 0, J_0 \geq 2 \vert 3)$		
spin	energy	isospin
$\frac{3}{2}$	$J_0 + \frac{5}{2}$	J_0
1	$J_0 + 3$	J_0 $J_0 - 1$
	$J_0 + 2$	$J_0 + 1$ J_0 $J_0 - 1$
$\frac{1}{2}$	$J_0 + \frac{7}{2}$	$J_0 - 1$
$\frac{1}{2}$	$J_0 + \frac{5}{2}$	$J_0 + 1$ J_0 J_0 $J_0 - 1$ $J_0 - 1$ $J_0 - 2$
$\frac{1}{2}$	$J_0 + \frac{3}{2}$	$J_0 + 1$ J_0 $J_0 - 1$
0	$J_0 + 3$	J_0 $J_0 - 1$ $J_0 - 2$
	$J_0 + 2$	$J_0 + 1$ J_0 J_0 $J_0 - 1$ $J_0 - 2$
	$J_0 + 1$	J_0

$SD(2, 0, 1 \vert 3)$		
spin	energy	isospin
$\frac{3}{2}$	$\frac{7}{2}$	1
1	4	1 0
	3	2 1 0
$\frac{1}{2}$	$\frac{9}{2}$	0
$\frac{1}{2}$	$\frac{7}{2}$	2 1 1 0
$\frac{1}{2}$	$\frac{5}{2}$	2 1 0
0	4	1
	3	2 1 1
	2	1

$SD(1, 0, 0 \vert 3)$		
spin	energy	isospin
$\frac{3}{2}$	$\frac{5}{2}$	0
1	2	1
$\frac{1}{2}$	$\frac{3}{2}$	1
0	2	0
	1	0

Table 2.15: The short $\mathcal{N} = 3$ gravitino multiplet: $E_0 = J_0 + \frac{3}{2}$. From left to right, $J_0 \geq 2, J_0 = 1$, and $J_0 = 0$ (that is massless).

$SD(J_0, 0, J_0 \geq 2\|3)$		
spin	energy	isospin
1	$J_0 + 1$	$\{\, J_0 - 1$
$\frac{1}{2}$	$J_0 + \frac{3}{2}$	$\begin{cases} J_0 - 1 \\ J_0 - 2 \end{cases}$
$\frac{1}{2}$	$J_0 + \frac{1}{2}$	$\begin{cases} J_0 \\ J_0 - 1 \end{cases}$
0	$J_0 + 2$	$\{\, J_0 - 2$
	$J_0 + 1$	$\begin{cases} J_0 \\ J_0 - 1 \\ J_0 - 2 \end{cases}$
	J_0	$\{\, J_0$

$SD(1, 0, 1\|3)$		
spin	energy	isospin
1	2	$\{\, 0$
$\frac{1}{2}$	$\frac{3}{2}$	$\begin{cases} 1 \\ 0 \end{cases}$
0	2	$\{\, 1$
	1	$\{\, 1$

$SD(1/2, 0, 1/2\|3)$		
spin	energy	isospin
$\frac{1}{2}$	1	$\{\, \frac{1}{2}$
0	$\frac{1}{2}$	$\{\, \frac{1}{2}$

Table 2.16: The $\mathcal{N} = 3$ vector multiplets: $E_0 = J_0$. The massive vector multiplet with $J_0 \geq 2$, the massless vector multiplet with $J_0 = 1$ and the supersingleton representation with $J_0 = 1/2$.

Chapter 3

The complete spectra of the $AdS_4 \times \left(\frac{G}{H}\right)_7$ solutions from harmonic analysis

In this chapter I consider the Freund Rubin solutions of eleven dimensional supergravity compactified on backgrounds

$$AdS_4 \times X_7 \tag{3.0.1}$$

in the three cases

$$\begin{aligned} X_7 &= M^{111} \qquad (\mathcal{N}=2) \\ X_7 &= Q^{111} \qquad (\mathcal{N}=2) \\ X_7 &= N^{010} \qquad (\mathcal{N}=3)\,. \end{aligned} \tag{3.0.2}$$

The complete mass spectra of the corresponding four dimensional supergravities are determined, by means of harmonic analysis. All the particles found fit into supermultiplets, and such supermultiplets are organized into UIRs of the flavour group G' (1.2.6). I stress that harmonic analysis enable us to solve this problem by means of group theory and differential geometry, without solving differential equations.

In section 1 I describe the Freund Rubin compactifications with $X_7 = G/H$ and discuss their symmetries. In section 2 I define and describe the M^{111} manifold and, in less detail, the Q^{111} and N^{010} manifolds. However, a further description of M^{111} and Q^{111} is given in the next chapter. In section 3 I review the theory of harmonic analysis on coset spaces, and how it can be applied to derive the mass spectra of Freund Rubin supergravities. In section 4 I describe the explicit derivation of the complete mass spectrum of $AdS_4 \times M^{111}$ supergravity. In section 5 I give the complete mass spectrum of $AdS_4 \times N^{010}$ supergravity, without describing its derivation by harmonic analysis; furthermore, I give a part of the mass spectrum of $AdS_4 \times Q^{111}$ supergravity, which has been found long ago without the help of harmonic analysis. Part of the content of the present chapter refers to results obtained within the collaborations [14], [17].

3.1 Supergravity on $AdS_4 \times G/H_7$

3.1.1 A summary of coset space differential geometry

Here I sketch very briefly the basic ideas of differential geometry on coset spaces, with some results that will be used afterwards. For the proof of these results and for a complete discussion of these topics, see [37], [58], [59].

Definitions

Let us consider a coset manifold G/H, whose dimension is $n = \mathrm{dim}G - \mathrm{dim}H$. It can be parametrized by the n coordinates y^α, on which the coset representatives $L(y)$ do depend. Under left multiplication by $g \in G$ we have:

$$gL(y) = L(y')\, h(y)\,. \tag{3.1.1}$$

The Lie algebra $\mathbb{G}$ of the group G admits the following orthogonal split:

$$\begin{gathered}\mathbb{G} = \mathbb{H} \oplus \mathbb{K}, \\ T_i \in \mathbb{H},\ T_a \in \mathbb{K},\ T_\Lambda \in \mathbb{G}\end{gathered} \tag{3.1.2}$$

where $\mathbb{H}$ contains the generators of H and $\mathbb{K}$ the remaining n generators. Then we can express the elements of G as $g = e^{y^a T_a} e^{x^i T_i}$, and the coset representatives as $L(y) = e^{y^a T_a}$.

We will consider *reductive* coset manifolds, namely, such that

$$[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}. \tag{3.1.3}$$

Furthermore, we will consider semisimple coset manifolds.

Since G/H is reductive, the n generators $T_a \in \mathbb{K}$ are in a representation of H, realized by means of the structure constants

$$C_{ia}{}^{b} = -\left(T_i^H\right)_a{}^b = \left(T_i^H\right)^b{}_a \tag{3.1.4}$$

(since by Jacobi identity $C_{ia}{}^{c} C_{jc}{}^{b} = \frac{1}{2} C_{ij}{}^{k} C_{ak}{}^{b}$). Being G semisimple we have $C_{ia}{}^{b} = C_{iab} = C_{i[ab]}$ (in a basis in which the Killing metric is the Kronecker delta), so the T^H are also $SO(n)$ generators in the fundamental representation. Hence,

$$H \subset SO(n) \tag{3.1.5}$$

and this embedding is realized by the generators (3.1.4). Notice that in the cases studied in this thesis we set $n = 7$.

Killing vectors

The transformation law $gL(y) = L(y')\, h(y)$ for infinitesimal g becomes

$$T_\Lambda L(y) = K_\Lambda(y)\, L(y) - L(y)\, T_i W_\Lambda^i(y) \tag{3.1.6}$$

where

$$\begin{aligned} g &= 1 + \epsilon^\Lambda T_\Lambda \\ h &= 1 - \epsilon^\Lambda W_\Lambda^i(y)\, T_i \\ y'^a &= y^a + \epsilon^\Lambda K_\Lambda^a(y)\,. \end{aligned} \tag{3.1.7}$$

The y–dependent matrices $W^i_\Lambda(y)$ are called *H–compensators*, and the y–dependent differential operators

$$K_\Lambda(y) \equiv K^a_\Lambda(y)\frac{\partial}{\partial y^a} \tag{3.1.8}$$

are called *Killing vectors on G/H*. We have

$$\begin{aligned}\left[T_\Lambda, T_\Sigma\right] L(y) &= C_{\Lambda\Sigma}{}^\Delta T_\Delta L(y)\,,\\ \left[K_\Lambda, K_\Sigma\right] &= -C_{\Lambda\Sigma}{}^\Delta K_\Delta.\end{aligned} \tag{3.1.9}$$

Vielbein, H–connection, H Lie derivative

The one–form

$$\Omega(y) = L^{-1}(y)\,dL(y) \tag{3.1.10}$$

is $\mathbb{G}$–valued, and can be expanded in a generator basis as follows:

$$\Omega(y) = \mathcal{B}^a(y)\,T_a + \Omega^i(y)\,T_i \tag{3.1.11}$$

where $\mathcal{B}^a(y) = \mathcal{B}_\alpha{}^a(y)\,dy^\alpha$ is a vielbein on G/H and $\Omega^i(y) = \Omega_\alpha{}^i(y)\,dy^\alpha$ is called the *H–connection*. Under left multiplication of an infinitesimal $g \in G$ this vielbein transforms as

$$\begin{aligned}\mathcal{B}^a(y+\delta y) &= \mathcal{B}^a(y) - \epsilon^\Lambda W^i_\Lambda(y)\,C_{ib}{}^a\mathcal{B}^b(y)\\ \delta y^a &= \epsilon^\Lambda K^a_\Lambda(y)\,.\end{aligned} \tag{3.1.12}$$

A vielbein transforming as above, namely, G–invariant modulo an H–compensator, is named a *G–left invariant vielbein*. Notice that the left action of G on $\mathcal{B}^a(y)$ is an H transformation in the fundamental representation of $SO(n)$. We can also define the metric on G/H,

$$g_{\alpha\beta}(y) = \gamma_{ab}\mathcal{B}^a_\alpha(y)\,\mathcal{B}^b_\beta(y) \tag{3.1.13}$$

(where γ_{ab} is the Killing metric of G restricted to G/H); it can be shown that this metric is left G–invariant, so G is an isometry of this metric; furthermore, this metric is insensitive to the choice of the coset parametrization.

The H–connection defines a parallel transport on the coset manifold, and then an H–covariant derivative

$$\mathcal{D}^H = d + \Omega^i T_i. \tag{3.1.14}$$

It can be written in terms of the embedding (3.1.4) $H \subset SO(n)$:

$$\mathcal{D}^H = d + \Omega^i\,(T_i)^{ab}\,t^{SO(n)}_{ab} \tag{3.1.15}$$

where $t^{SO(n)}_{ab}$ are the $SO(n)$ generators. For example, for the vector representation they are $\left(t^{SO(n)}_{ab}\right)^{cd} = -\mathrm{i}\delta^{cd}_{ab}$, while for the spinor representation they are given by the two-indices gamma matrices.

Let us determine the action of the H–covariant derivative on the inverse of the coset representative. Being

$$LdL^{-1} = -dLL^{-1}, \tag{3.1.16}$$

we have

$$\Omega L^{-1} = L^{-1}dLL^{-1} = -dL^{-1} = \left(\Omega^i T_i + \mathcal{B}^a T_a\right) L^{-1}, \tag{3.1.17}$$

then

$$\mathcal{D}^H L^{-1} = \left(d + \Omega^i T_i\right) L^{-1} = -\mathcal{B}^a T_a L^{-1}. \tag{3.1.18}$$

This action is purely algebraic. As we will see, such a property is the core of the harmonic analysis method for solving differential equations.

The Lie derivative associated to a Killing vector, acting on the vielbein, is

$$l_{K_\Lambda} \mathcal{B}^a\left(y\right) = W^i_\Lambda\left(y\right) C_{ib}{}^a \mathcal{B}^b\left(y\right). \tag{3.1.19}$$

Then, if we define the *H–covariant Lie derivative*

$$\mathcal{L}_{K_\Lambda} \equiv l_{K_\Lambda} - W^i_\Lambda\left(y\right) T_i, \tag{3.1.20}$$

which satisfies all the properties of the Lie derivative, we have

$$\mathcal{L}_{K_\Lambda} \mathcal{B}^a\left(y\right) = 0. \tag{3.1.21}$$

On the coset representative the action of the H–covariant Lie derivative is

$$\mathcal{L}_{K_\Lambda} L\left(y\right) = T_\Lambda L\left(y\right). \tag{3.1.22}$$

Spin connection

Another useful structure we can build on our coset manifold is a Riemannian connection $\mathcal{B}^a{}_b$, or *spin connection*, that defines a parallel transport. It is an $so\left(n\right)$–valued one–form defined by the vanishing torsion equation

$$\mathcal{R}^a \equiv d\mathcal{B}^a - \mathcal{B}^{ab} \wedge \mathcal{B}_b = 0. \tag{3.1.23}$$

The Riemannian curvature $\mathcal{R}^a{}_b$ is an $so\left(n\right)$–valued two–form defined by

$$\mathcal{R}^{ab} \equiv d\mathcal{B}^{ab} - \mathcal{B}^{ac} \wedge \mathcal{B}_c{}^b = \mathcal{R}^{ab}{}_{cd} \mathcal{B}^c \wedge \mathcal{B}^d. \tag{3.1.24}$$

Notice that in this way we define a parallel transport by $SO\left(n\right)$ transformations on the vielbein; in other words, we select the $SO\left(n\right)$ group as the tangent group. The vielbein is then in the vector $SO\left(n\right)$ representation, and all the fields on the manifold are in $SO\left(n\right)$ representations. Being $H \subset SO\left(n\right)$, the fields in irreducible representations of $SO\left(n\right)$ can be branched in fields in irreducible representations of H.

Expanding the spin connection one finds [1]

$$\mathcal{B}^a{}_b = -C_{ib}{}^a \Omega^i + \frac{1}{2} C_{bc}{}^a \mathcal{B}^c = -\left(T^H_i\right)^a{}_b \Omega^i + \frac{1}{2} C_{bc}{}^a \mathcal{B}^c. \tag{3.1.25}$$

The first term in this expression is valued in $\mathbb{H} \subset so\left(n\right)$ (whose generators are the $C_{bi}{}^a$), while the last term is valued in other $so\left(n\right)$ generators. In other words, the spin connection contains the H–connection plus other $so\left(n\right)$–valued terms.

[1] in the following we call T^H_i the generators of $\mathbb{H}$ and T^K_a the generators of $\mathbb{K}$, for clarity of notations

The spin connection naturally defines a $SO(n)$ covariant derivative

$$\mathcal{D}^{SO(n)} = d - \mathcal{B}^{ab} t^{SO(n)}_{ab}. \tag{3.1.26}$$

Substituting the (3.1.25), we find an expression of the form

$$\begin{aligned} \mathcal{D}^{SO(n)} &= d + \left(T^H_i\right)^{ab} \Omega^i t^{SO(n)}_{ab} + \frac{1}{2} C_{bc}{}^a \mathcal{B}^c t^{SO(n)}{}_a{}^b = \\ &= \mathcal{D}^H + \mathbb{M}_c \mathcal{B}^c. \end{aligned} \tag{3.1.27}$$

A very useful property of the $SO(n)$ covariant derivative is that it commutes with the H–covariant Lie derivative:

$$\left[\mathcal{L}_{K_\Lambda}, \mathcal{D}^{SO(n)}\right] = 0. \tag{3.1.28}$$

It follows that, because of the Schur's lemma, $\mathcal{D}^{SO(n)}$ acts irreducibly on G representations, namely, it cannot change a representation of G in another one.

Rescalings

In general the metric (3.1.13) is not the only G–invariant metric on G/H; it is unique only up to some particular *rescalings* of the vielbein

$$\mathcal{B}^a = r^a \mathcal{B}'^a \quad \text{no sum on } a. \tag{3.1.29}$$

The (3.1.25) changes with rescalings by coefficients depending on the r^a's, and the same happens to the matrices $\mathbb{M}$, but the properties (3.1.18), (3.1.28) remain satisfied. By means of these rescalings, it is sometimes possible to obtain an Einstein metric, namely, a metric such that

$$\mathcal{R}^a{}_b = \Lambda \delta^a_b, \tag{3.1.30}$$

even if the non–rescaled metric is non–Einstein. Furthermore, by a global vielbein rescaling one can choose the value of Λ.

In the following, to avoid confusion between the not rescaled vielbein and the rescaled one, we call Ω^a the not rescaled vielbein, namely,

$$\Omega \equiv L^{-1} dL = \Omega^i T^H_i + \Omega^a T^K_a \tag{3.1.31}$$

and $\mathcal{B}^a$ the rescaled vielbein

$$\mathcal{B}_a = \frac{1}{r_a} \Omega_a. \tag{3.1.32}$$

So, for example, the (3.1.18) becomes

$$\mathcal{D}^H L^{-1} = -\Omega^a T^K_a L^{-1} = -r_a \mathcal{B}^a T^K_a L^{-1}, \tag{3.1.33}$$

or, expanding on the vielbein $\mathcal{D}^H = \mathcal{B}^a \mathcal{D}^H_a$,

$$\mathcal{D}^H_a L^{-1} = -r_a T_a L^{-1}. \tag{3.1.34}$$

3.1.2 The Freund Rubin solution

As I said in chapter 1, given a seven dimensional compact coset manifold G/H, the (1.2.13) is a solution of eleven dimensional supergravity, with the geometry of $AdS_4 \times G/H$, and can be viewed as a four dimensional anti–de Sitter supergravity with internal space G/H.

Let us express this in the formalism of rheonomy (for a review on rheonomy, see [37]). We use here the following conventions:

$$\begin{array}{rc} m, n & \text{flat indices on } AdS_4 \\ a, b & \text{flat indices on } G/H \\ \mu\nu & \text{curved indices on } AdS_4 \\ \alpha, \beta & \text{curved indices on } G/H \\ \hat{a}, \hat{b} & \text{eleven dimensional flat indices} \\ M, N & SO(\mathcal{N}) \text{ indices} \\ x^\mu & \text{coordinates on } AdS_4 \\ y^\alpha & \text{coordinates on } G/H \\ \theta & \text{fermionic coordinates in } AdS_4 \text{ superspace.} \end{array} \tag{3.1.35}$$

We call τ_a the $SO(7)$ gamma matrices, which are 8×8 and act on the G/H spinors (which are in the spinor representation of $SO(7)$):

$$\{\tau_a, \tau_b\} = 2\eta_{ab} = 2\ \mathrm{diag}\,(-,-,-,-,-,-,-)\,. \tag{3.1.36}$$

We suppose that it is possible to define an Einstein metric structure on G/H such that

$$\mathcal{R}^{ac}{}_{bc} = 12e^2\delta^a_b. \tag{3.1.37}$$

The $SO(7)$ generators in the spinor representation are

$$t^{SO(7)}_{ab} = \frac{1}{4}\tau_{ab} = \frac{1}{8}\left[\tau_a, \tau_b\right]. \tag{3.1.38}$$

We call $\mathcal{N}$ the number of independent $SO(7)$ real spinors $\eta_M(y)$ satisfying the equation

$$\mathcal{D}^{SO(7)}\eta_M = \left(d - \frac{1}{4}\mathcal{B}^{ab}\tau_{ab}\right)\eta_M = e\mathcal{B}^a\tau_a\eta_M. \tag{3.1.39}$$

Notice that the η_M, which we call *Killing spinors on* G/H, are made of $\mathbb{C}$–numbers, not of grassmannian variables, and are then commuting.

Let us consider now the $\mathcal{N}$ extended AdS_4 supergravity. Its superspace is

$$\mathcal{M}_{4\mathcal{N}|4} = \frac{Osp(\mathcal{N}|4)}{SO(1,3)\times SO(\mathcal{N})}. \tag{3.1.40}$$

Let $\overset{o}{V}{}^m(x,\theta)$, $\overset{o}{\omega}{}^{mn}(x,\theta)$, $\overset{o}{A}{}^{MN}(x,\theta)$, $\overset{o}{\psi}_M(x,\theta)$ be the left–invariant one forms on $\mathcal{M}_{4\mathcal{N}|4}$. They fulfill by definition the following Maurer Cartan equations

$$\begin{aligned} d\overset{o}{V}{}^m - \overset{o}{\omega}{}^m{}_n \wedge \overset{o}{V}{}^n - \frac{1}{2}\mathrm{i}\,\overset{o}{\bar\psi}_M \wedge\gamma^m \overset{o}{\psi}_M &= 0 \\ d\overset{o}{\omega}{}^{mn} - \overset{o}{\omega}{}^{mr} \wedge \overset{o}{\omega}{}_r{}^n + 16e^2\overset{o}{V}{}^m \wedge \overset{o}{V}{}^n - 2\mathrm{i}e^2\,\overset{o}{\bar\psi}_M \wedge\gamma_5\gamma^{mn}\overset{o}{\psi}_M &= 0 \\ d\overset{o}{A}{}^{MN} + e\overset{o}{A}{}^{MR} \wedge \overset{o}{A}{}_R{}^N - 4\mathrm{i}\,\overset{o}{\bar\psi}_M \wedge\gamma_5\overset{o}{\psi}_N &= 0 \\ d\overset{o}{\psi}_M - \frac{1}{4}\gamma^{mn}\overset{o}{\omega}_{mn}\overset{o}{\psi}_M - e\overset{o}{A}{}_M{}^N \wedge \overset{o}{\psi}_N - 2e\gamma_5\gamma_m\overset{o}{V}{}^m \wedge \overset{o}{\psi}_M &= 0. \end{aligned} \tag{3.1.41}$$

With all these objects we can build the Freund Rubin solution of eleven dimensional supergravity: the following eleven dimensional forms [2]

$$V^{\hat{a}} = (V^m, B^a)\,,\ \omega^{\hat{a}\hat{b}} = \left(\omega^{mn}, K^{ma}, B^{ab}\right) \tag{3.1.42}$$

$$\begin{aligned}
V^m &= \overset{o}{V}{}^m (x,\theta) \\
\omega^{mn} &= \overset{o}{\omega}{}^{mn} (x,\theta) \\
\psi &= \overset{o}{\psi}_M (x,\theta)\, \eta^M (y) \\
B^a &= \mathcal{B}^a (y) + \frac{1}{8}\bar{\eta}_M (y)\, \tau^a \eta_N (y)\ \overset{o}{A}{}^{MN} (x,\theta) \\
B^{ab} &= \mathcal{B}^{ab} (y) - \frac{1}{4} e\bar{\eta}_M (y)\, \tau^{ab} \eta_N (y)\ \overset{o}{A}{}^{MN} (x,\theta) \\
K^{ma} &= 0
\end{aligned} \tag{3.1.43}$$

and $A = \overset{o}{A} (x,y,\theta)$ three-form not globally defined (it is a section of a fiber bundle), such that

$$\begin{aligned}
d\,\overset{o}{A} \;=\;& e\epsilon_{mnrs}\, \overset{o}{V}{}^m \wedge \overset{o}{V}{}^n \wedge \overset{o}{V}{}^r \wedge \overset{o}{V}{}^s + \frac{1}{2}\, \bar{\overset{o}{\psi}}_M \wedge \gamma^{mn}\, \overset{o}{\psi}{}^M \wedge \overset{o}{V}{}^m \wedge \overset{o}{V}{}^n + \\
& - \bar{\overset{o}{\psi}}_M \wedge \gamma_5 \gamma_m\, \overset{o}{\psi}_N \wedge \overset{o}{V}{}^m \wedge \bar{\eta}_M \tau_a \eta_N B^a + \frac{1}{2}\, \bar{\overset{o}{\psi}}_M \wedge \overset{o}{\psi}_N\, \bar{\eta}_M \tau_{ab} \eta_N B^a \wedge B^b,
\end{aligned} \tag{3.1.44}$$

satisfy the Maurer Cartan equations of eleven dimensional supergravity.

If we evaluate these superspace forms on a bosonic surface, i.e. at $\theta = 0$, we get the fields of the $\mathcal{N}$–extended eleven dimensional Freund Rubin solution of supergravity:

$$\begin{array}{rclrcl}
g_{\mu\nu} (x,y) &=& g^0_{\mu\nu} (x) & F_{\mu\nu\rho\sigma} &=& e\sqrt{g^0}\varepsilon_{\mu\nu\rho\sigma} \\
g_{\alpha\beta} (x,y) &=& g^0_{\alpha\beta} (y) & \text{other } F &=& 0 \\
g_{\mu\alpha} &=& 0 & \psi_\mu = \psi_\alpha &=& 0
\end{array} \tag{3.1.45}$$

where $g^0_{\mu\nu}$ is the AdS_4 metric, $g^0_{\alpha\beta}$ is the G–invariant G/H metric.

This solution preserves $\mathcal{N}$ supersymmetries. In fact, being $\psi_{\hat{a}} (x,y) = 0$, the supersymmetry transformations of the bosonic fields vanish. The supersymmetry transformations of the gravitino fields are:

$$\begin{aligned}
\delta_\epsilon \psi_\mu (x,y) &= \left(\partial_\mu - \frac{1}{4}\omega^{mn}_\mu \gamma_{mn} + 2e\gamma_5\gamma_\mu V^m_\mu\right) \epsilon (x,y) \\
\delta_\epsilon \psi_\alpha (x,y) &= \left(\partial_\alpha - \frac{1}{4}\mathcal{B}^{ab}_\alpha \tau_{ab} - e\tau_a \mathcal{B}^a_\alpha\right) \epsilon (x,y)\,.
\end{aligned} \tag{3.1.46}$$

They vanish for

$$\epsilon (x,y) = \epsilon (x)\, \eta (y) \tag{3.1.47}$$

[2] We leave implicit the spinor indices; remind that a four dimensional AdS_4 spinor has an index taking four values, a seven dimensional G/H spinor has an index taking eight values, the eleven dimensional spinor has an index taking thirty-two values, and in fact the tensor product of an AdS_4 spinor and a G/H spinor is an eleven dimensional spinor.

where $\epsilon(x)$ is an AdS_4 Killing spinor, satisfying

$$\left(\partial_m - \frac{1}{4}\omega_m^{rs}\gamma_{rs} + 2e\gamma_5\gamma_m\right)\epsilon(x) = 0 \tag{3.1.48}$$

and $\eta(y)$ is a G/H Killing spinor, satisfying the (3.1.39). There are four independent solutions of the (3.1.48), and, as I said, $\mathcal{N}$ is the number of the independent solutions of the (3.1.39), then there exist $4\mathcal{N}$ independent one–component supersymmetry transformations leaving invariant the (3.1.45). With the conventions of four dimensional supergravity (where supercharges have four pseudo–real components), this means that the solution preserves $\mathcal{N}$ supersymmetries.

Notice that the Freund Rubin solution is a *spontaneous compactification*, in the sense that $AdS_4 \times X_7$ is a solution of eleven dimensional supergravity, and then an allowed vacuum around which we can perform perturbation theory; nothing has been added to the theory at hand. The key that allows this is the presence of a four–form field strength, to which we can give the expectation value of ϵ_{mnrs}, the invariant tensor of $SO(1,3)$, breaking $11 \longrightarrow 4+7$.

The solution of the G/H Killing spinor equation and holonomy

Given a coset manifold G/H admitting an Einstein metric structure, we want to know if the corresponding Freund Rudin solution is supersymmetric, and how much. As we have seen, this can be done by solving the G/H Killing spinor equation (3.1.39).

First of all we have to consider the integrability conditions of the (3.1.39). They are

$$\mathcal{C}_{ab}\eta \equiv \left(\mathcal{R}^{cd}{}_{ab} - 4e^2\delta^{cd}_{ab}\right)\tau_{cd}\eta = \mathcal{C}_{ab}{}^{cd}\tau_{cd}\eta. \tag{3.1.49}$$

We have to find the null eigenspinors of the 21 $\mathcal{C}_{ab}$ operators here defined, which are combination of the τ_{ab} generators with coefficients $\mathcal{C}_{ab}{}^{cd}$ (which are the components of the Weyl tensor) and then generate a subgroup of $SO(7)$. Being the 8 dimensional spinor representation of $SO(7)$ irreducible, the equation $\tau_{ab}\eta = 0$ has no solutions, and then the (3.1.49) has null eigenspinors only if the combinations $\mathcal{C}_{ab}$ do not generate all $SO(7)$ but lie, with their commutators, in a subspace of the $SO(7)$ algebra, under which the 8 dimensional spinor representation of $SO(7)$ be reducible. This algebra is called the *Weyl holonomy algebra* $\mathcal{G}_{hol}$. It is slightly different from the usual holonomy algebra of riemannian geometry, namely, the algebra of transformations that can occur to a vector after parallel riemannian transport around a closed curve, which is the Riemann holonomy algebra; the latter is generated by the Riemann tensor, not by the Weyl tensor (see [58]); $\mathcal{G}_{hol}$ is the holonomy algebra with respect to the parallel transport defined by the covariant derivative $\mathcal{D}^{SO(7)} - e\mathcal{B}^a\tau_a$. So, for example, the Riemannian holonomy algebra of S^7 is $SO(7)$, while the Weyl holonomy algebra of S^7 is $\{0\}$. Notice that the generators in the (3.1.39), (τ_{ab}, τ_a), are the generators of $SO(8)$; the Killing spinors, in fact, are covariantly constant under an $SO(8) \supset SO(7)$ group.

So if $\mathcal{G}_{hol} = SO(7)$, $\mathcal{N}_{MAX} = 0$. If $\mathcal{G}_{hol} = G_2$, being

$$\mathbf{8} \overset{G_2 \subset SO(7)}{\longrightarrow} \mathbf{7} \oplus \mathbf{1}, \tag{3.1.50}$$

$\mathcal{N}_{MAX} = 1$. If $\mathcal{G}_{hol} = SU(3)$, being

$$\mathbf{8} \overset{SU(3) \subset SO(7)}{\longrightarrow} \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{3.1.51}$$

$\mathcal{N}_{MAX} = 2$. If $\mathcal{G}_{hol} = SU(2)$, being

$$\mathbf{8} \overset{SU(2)\subset SO(7)}{\longrightarrow} \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{3.1.52}$$

$\mathcal{N}_{MAX} = 4$. If $\mathcal{G}_{hol} = \{0\}$, $\mathcal{N}_{MAX} = 8$.

Then, in order to find the solutions of the (3.1.39), one has to find the holonomy group, then to determine the null eigenspinors of the integrability condition $C_{ab}(y)\,\eta(y) = 0$, and finally substitute these eigenspinors in the (3.1.39) to check if they are actually solutions.

From Killing spinors to Killing vectors

There is an interesting property of G/H Killing spinors. Given the $\mathcal{N}$ Killing spinors $\eta_M(y)$, namely, the solutions of the (3.1.39), we can build the following $\mathcal{N}(\mathcal{N}-1)/2$ vectors on G/H

$$k^a_{MN}(y) \equiv \bar{\eta}_{[M}\tau^a\eta_{N]}. \tag{3.1.53}$$

It can be shown that these are Killing vectors of G/H, generating an $SO(\mathcal{N})$ group which is then an isometry of the coset manifold. This is the reason of the previously stressed property

$$G = G' \times SO(\mathcal{N}). \tag{3.1.54}$$

3.1.3 Four dimensional supergravity from Freund Rubin solution

Given a Freund Rubin solution of eleven dimensional supergravity, we can consider this classical solution as a vacuum of the theory, and do perturbation theory taking as dynamical degrees of freedom the fluctuation around this vacuum (see [20], [37]):

$$\begin{aligned}
g_{\mu\nu}(x,y) &= g^0_{\mu\nu}(x) + h_{\mu\nu}(x,y)\\
g_{\alpha\beta}(x,y) &= g^0_{\alpha\beta}(x) + h_{\alpha\beta}(x,y)\\
g_{\mu\alpha}(x,y) &= h_{\mu\alpha}(x,y)\\
A_{\mu\nu\rho}(x,y) &= A^0_{\mu\nu\rho}(x) + a_{\mu\nu\rho}(x,y)\\
A_{\mu\nu\alpha}(x,y) &= a_{\mu\nu\alpha}(x,y)\\
A_{\mu\alpha\beta}(x,y) &= a_{\mu\alpha\beta}(x,y)\\
A_{\alpha\beta\gamma}(x,y) &= a_{\alpha\beta\gamma}(x,y)\ .
\end{aligned} \tag{3.1.55}$$

The equations of eleven dimensional supergravity, linearized in these fluctuations, have in general the form

$$\left(\square_x^{[E\,s]} + \boxtimes_y^{[\lambda_1\lambda_2\lambda_3]}\right)\Phi^{[E\,s]}_{[\lambda_1\lambda_2\lambda_3]}(x,y) = 0. \tag{3.1.56}$$

Here $\Phi^{[E\,s]}_{[\lambda_1\lambda_2\lambda_3]}(x,y)$ is a field transforming in the irreducible representation $[E\,s]$ of $SO(3,2)$ and $[\lambda_1\lambda_2\lambda_3]$ of $SO(7)$ [3], and depends both on the coordinates x of anti–de Sitter space and on the coordinates y of G/H. Notice that Φ has $SO(7)$ indices because, as I explained, a generic field on G/H is in an irreducible representation of $SO(7)$. $\square_x^{[E\,s]}$ is the kinetic operator for a field of energy and spin $[E\,s]$ on AdS_4, and is well known from AdS_4 theory

[3] $[\lambda_1\lambda_2\lambda_3]$ are the Dynkin labels of the $SO(7)$ UIR ($SO(7)$ has rank three).

(see chapter 2). $\boxtimes_y^{[\lambda_1\lambda_2\lambda_3]}$ is the kinetic operator for a field of spin $[\lambda_1\lambda_2\lambda_3]$ on the seven dimensional G/H. The operators $\boxtimes_y^{[\lambda_1\lambda_2\lambda_3]}$ are built with the $SO(7)$–covariant derivative $\mathcal{D}^{SO(7)}$, the Killing metric on G/H, and, for spinor fields, the gamma matrices τ_a. They all have the property of the $SO(7)$–covariant derivative to be *invariant operators*, namely, to commute with the H–invariant Lie derivative:

$$\left[\boxtimes_y^{[\lambda_1\lambda_2\lambda_3]}, \mathcal{L}_{K_A}\right] = 0. \tag{3.1.57}$$

As I explain in section 3.3, we can expand the field Φ in a complete set of eigenfunctions of $\boxtimes_y$, the G/H harmonics:

$$\Phi(x,y) = \sum \mathcal{H}(y)\,\phi(x) \tag{3.1.58}$$

$$\boxtimes_y \mathcal{H}(y) = M\mathcal{H}(y). \tag{3.1.59}$$

The differential equation (3.1.56) becomes

$$(\Box_x + M)\,\phi(x) = 0 \tag{3.1.60}$$

which is an equation for a four dimensional supergravity field on AdS_4.

Then the eleven dimensional supergravity linearized around the Freund Rubin solution looks like the **$\mathcal{N}$–extended four dimensional supergravity on** AdS_4.

The explicit expression of the expansion (3.1.59) of the fields (3.1.55) is

$$\begin{aligned}
h_{mn}(x,y) &= \Big(h^I_{mn}(x) - \frac{3}{M_{(0)^3}+32}\mathcal{D}_{(m}\mathcal{D}_{n)}\Big[(2+\sqrt{M_{(0)^3}+36}\,)S^I(x) + \\
&\quad +(2-\sqrt{M_{(0)^3}+36}\,)\Sigma^I(x)\Big] + \tfrac{5}{4}\delta_{mn}\Big[(6-\sqrt{M_{(0)^3}+36}\,)S^I(x) + \\
&\quad +(6+\sqrt{M_{(0)^3}+36}\,)\Sigma^I(x)\Big]\Big)\mathcal{Y}^I(y)\ , \\
h_{ma}(x,y) &= [(\sqrt{M_{(1)(0)^2}+16}-4)A^I_m(x) + (\sqrt{M_{(1)(0)^2}+16}+4)W^I_m(x)\,]\,\mathcal{Y}^I_a(y)\ , \\
h_{ab}(x,y) &= \phi^I(x)\,\mathcal{Y}^I_{(ab)}(y) - \delta_{ab}\Big[(6-\sqrt{M_{(0)^3}+36})S^I(x) + \\
&\quad +(6+\sqrt{M_{(0)^3}+36})\Sigma^I(x)\Big]\ \mathcal{Y}^I(y)\ , \\
a_{mnr}(x,y) &= 2\,\varepsilon_{mnrp}\,\mathcal{D}_p(S^I(x)+\Sigma^I(x))\mathcal{Y}^I(y)\ , \\
a_{mna}(x,y) &= \tfrac{2}{3}\,\varepsilon_{mnrs}\,(\mathcal{D}_r A^I_s(x) + \mathcal{D}_r W^I_s(x))\,\mathcal{Y}^I_a(y)\ , \\
a_{mab}(x,y) &= Z^I_m(x)\,\mathcal{Y}^I_{[ab]}(y)\ , \\
a_{abc}(x,y) &= \pi^I(x)\,\mathcal{Y}^I_{[abc]}(y)\ , \\
\psi_m(x,y) &= \Big(\chi^I_m(x) + \frac{\frac{4}{7}M_{(1/2)^3}+8}{M_{(1/2)^3}+8}[D_m\lambda^I_L(x)\,]_{3/2} -
\end{aligned}$$

$$+(6+\tfrac{3}{7}M_{(1/2)^3})\gamma_5\gamma_m\lambda_L^I(x)\Big)\,\Xi^I(y)\;,$$

$$\psi_a \;=\; \lambda_T^I(x)\,\Xi_a^I(y)+\lambda_L^I(x)\,[\nabla_a\Xi^I(y)]_{3/2}\,. \tag{3.1.61}$$

The conventions for the names of the harmonics $\mathcal{H}^I$ and their eigenvalues are the following:

$SO(7)$ UIR	Harmonic $\mathcal{H}$	Eigenvalue $M_{[\lambda_1,\lambda_2,\lambda_3]}$
$[0,0,0]$	$\mathcal{Y}$	$M_{(0)^3}$
$[1,0,0]$	$\mathcal{Y}_a, \quad \mathcal{D}^a\mathcal{Y}_a=0$	$M_{(1)(0)^2}$
$[1,1,0]$	$\mathcal{Y}_{[ab]}, \quad \mathcal{D}^a\mathcal{Y}_{[ab]}=0$	$M_{(1)^2(0)}$
$[1,1,1]$	$\mathcal{Y}_{[abc]}, \quad \mathcal{D}^a\mathcal{Y}_{[abc]}=0$	$M_{(1)^3}$
$[2,0,0]$	$\mathcal{Y}_{(ab)}, \quad \eta^{ab}\mathcal{Y}_{(ab)}=\mathcal{D}^a\mathcal{Y}_{(ab)}=0$	$M_{(2)(0)^2}$
$\left[\frac{1}{2},\frac{1}{2},\frac{1}{2}\right]$	Ξ	$M_{\left(\frac{1}{2}\right)^3}$
$\left[\frac{3}{2},\frac{1}{2},\frac{1}{2}\right]$	$\Xi_a, \quad \tau^a\Xi_a=\mathcal{D}^a\Xi_a=0$	$M_{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2}$

(3.1.62)

I explain in section 3.3 how are defined the harmonics and why in the expansion (3.1.61) they have an index I, running in an UIR of G.

To each of these $SO(7)$ UIRs does correspond an invariant operator on G/H arising from linearization of the eleven dimensional supergravity equations. They are the following (we call $\mathcal{D}\equiv\mathcal{D}^{SO(7)}$):

- 0–form Hodge de Rahm operator (the Laplacian)

$$\boxtimes_y^{[0,0,0]}\mathcal{Y}\equiv\mathcal{D}^a\mathcal{D}_a\mathcal{Y}=M_{(0)^3}\mathcal{Y}. \tag{3.1.63}$$

- 1–form Hodge de Rahm operator

$$\boxtimes_y^{[1,0,0]}\mathcal{Y}^a=\left(\mathcal{D}^a\mathcal{D}_a+24e^2\right)\mathcal{Y}^a=M_{(1)(0)^2}\mathcal{Y}^a. \tag{3.1.64}$$

- 2–form Hodge de Rahm operator

$$\begin{aligned}\boxtimes_y^{[1,1,0]}\mathcal{Y}^{[ab]} &= \left(\mathcal{D}^a\mathcal{D}_a+48e^2\right)\mathcal{Y}^{[ab]}+\\ &\quad -4\mathcal{R}^{[a\ \ b]}_{\ \ [c\ \ d]}\mathcal{Y}^{[cd]}=M_{(1)^2(0)}\mathcal{Y}^{[ab]}.\end{aligned} \tag{3.1.65}$$

- 3–form first order operator

$$\begin{aligned}\boxtimes_y^{[1,1,1]}\mathcal{Y}^{[abc]} &= \frac{1}{24}\epsilon^{abcd}{}_{efg}\mathcal{D}_d\mathcal{Y}^{efg}=\\ &= M_{(1)^3}\mathcal{Y}^{[abc]}.\end{aligned} \tag{3.1.66}$$

- Lichnerowicz operator

$$\begin{aligned}\boxtimes_y^{[2,0,0]}\mathcal{Y}^{(ab)} &= \mathcal{D}^c\mathcal{D}_c\mathcal{Y}^{(ab)}+4\mathcal{R}^{a\ \ b}_{\ \ c\ \ d}\mathcal{Y}^{(cd)}+\\ &\quad +2\mathcal{R}^a{}_c\mathcal{Y}^{(bc)}+2\mathcal{R}^b{}_c\mathcal{Y}^{(ac)}=M_{(2)(0)^2}\mathcal{Y}^{(ab)}.\end{aligned} \tag{3.1.67}$$

- Dirac operator

$$\boxtimes_y^{[1/2,1/2,1/2]}\Xi=\left(\tau^a\mathcal{D}_a-7e\right)\Xi=M_{(1/2)^3}\Xi. \tag{3.1.68}$$

- Rarita Schwinger operator

$$\boxtimes_y^{[3/2,1/2,1/2]}\Xi_a = \left(\tau^a\mathcal{D}_a - 5e\right)\Xi_a = M_{(3/2)(1/2)^2}\Xi. \tag{3.1.69}$$

The AdS_4 fields appearing in the expansion (3.1.61) are the following:

- one spin 2 field $h_{mn}(x)$, arising from the expansion of the eleven dimensional graviton along the AdS_4 directions;
- two spin 1 fields, $A_m(x)\,,\ W_m(x)$, arising from the expansions of the components $h_{ma}(x,y)$ of the eleven dimensional graviton, and from the components a_{mna} of the three form; as the massless graviton gauges the symmetries in eleven dimensional supergravity, the massless vectors A_m gauge the isometry G;
- one spin 1 field $Z_m(x)$, arising from the expansion of the components a_{mab} of the eleven dimensional three form; in the (3.1.61) it is the coefficient of a two form G/H harmonic $\mathcal{Y}_{[ab]}$; there is one massless Z_m field for each harmonic two form $\mathcal{Y}_{[ab]}$ on G/H, then the massless Z_m are counted by the second Betti number b_2 of G/H;
- two scalar fields $S(x)\,,\ \Sigma(x)$, arising from the expansion of the graviton and of the components a_{mnr} of the three form;
- one scalar field $\phi(x)$, arising from the expansion of the graviton along the G/H directions;
- one pseudo–scalar field $\pi(x)$ arising from the expansion of the components a_{abc} of the three form;
- two spinor fields $\lambda_L(x)\,,\ \lambda_T(x)$ arising from the expansion of the eleven dimensional gravitino;
- one gravitino field χ_m arising from the expansion of the eleven dimensional gravitino along the AdS_4 directions.

Substituting the harmonic expansion (3.1.61) of the eleven dimensional fields and the (3.1.63),. . .,(3.1.69) eigenvalue equations into the linearized equation of supergravity (3.1.56), one finds equations for the AdS_4 fields with masses given by the G/H harmonic eigenvalues $M_{(0)^3}, \ldots, M_{(3/2)(1/2)^2}$. One finds [20]:

$$\begin{aligned}
m_h^2 &= M_{(0)^3}\,, \\
m_\Sigma^2 &= M_{(0)^3} + 176 + 24\sqrt{M_{(0)^3}+36}\,, \\
m_S^2 &= M_{(0)^3} + 176 - 24\sqrt{M_{(0)^3}+36}\,, \\
m_\phi^2 &= M_{(2)(0)^2}\,, \\
m_\pi^2 &= 16\left(M_{(1)^3}-2\right)\left(M_{(1)^3}-1\right)\,, \\
m_W^2 &= M_{(1)(0)^2} + 48 + 12\sqrt{M_{(1)(0)^2}+16}\,, \\
m_A^2 &= M_{(1)(0)^2} + 48 - 12\sqrt{M_{(1)(0)^2}+16}\,,
\end{aligned}$$

$$
\begin{aligned}
m_Z^2 &= M_{(1)^2(0)}\,, \\
m_{\lambda_L} &= -\left(M_{\left(\frac{1}{2}\right)^3}+16\right)\,, \\
m_{\lambda_T} &= M_{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)^2}+8\,, \\
m_\chi &= M_{\left(\frac{1}{2}\right)^3}\,.
\end{aligned} \tag{3.1.70}
$$

I remind that the masses of AdS_4 fields are related to their energies by the (2.2.14).

Summarizing, if we want to find the mass spectrum of a four dimensional supergravity obtained by Freund Rubin compactification with a coset manifold G/H, we have to determine the spectrum of the invariant operators (3.1.63),. . .,(3.1.69) on the coset manifold; from this, by the mass formula (3.1.70), we can find all the masses of the AdS_4 fields in the supergravity. Looking at the expansion (3.1.61), we see that:

- for each eigenvalue of the zero–form harmonic $\mathcal{Y}(y)$ there are one graviton field $h_{mn}(x)$, one scalar field $S(x)$ and one scalar field $\Sigma(x)$;
- for each eigenvalue of the one–form harmonic $\mathcal{Y}^a(y)$ there are one vector field $A_m(x)$ and one vector field $W_m(x)$;
- for each eigenvalue of the two–form harmonic $\mathcal{Y}^{[ab]}(y)$ there is one vector field $Z_m(x)$;
- for each eigenvalue of the three–form harmonic $\mathcal{Y}^{[abc]}(y)$ there is one pseudo–scalar field $\pi(x)$;
- for each eigenvalue of the harmonic $\mathcal{Y}^{(ab)}(y)$ there is one scalar field $\phi(x)$;
- for each eigenvalue of the spinor harmonic $\Xi(y)$ there is one spinor field $\lambda_L(x)$ (called the longitudinal spinor field) and one gravitino field $\chi_m(x)$;
- for each eigenvalue of the spinor–vector harmonic $\Xi_a(y)$ there is one spinor field $\lambda_T(x)$ (called the transverse spinor field).

3.1.4 Supersymmetric mass relations

A useful tool for deriving the mass spectrum of a Freund Rubin supergravity are the *supersymmetric mass relations* [22], [37]. The key point is that it is possible to build G/H harmonics eigenfunctions of invariant operators by means of other G/H harmonics eigenfunctions of other invariant operators. The eigenvalues of these harmonics are related, and then for each eigenvalue of the latter invariant operator there is one eigenvalue of the former, given by relations which can be worked out. Then, using the (3.1.70), one can translate these relations between G/H–harmonics eigenvalues into relations between AdS_4 fields masses.

These relations can be understood from a different point of view. The different fields of $\mathcal{N}$–extended AdS_4 supergravity are related by supersymmetry transformations. While in Poincarè supersymmetry the fields in a same supermultiplet have the same mass, in AdS_4 supersymmetry it is not so (see chapter 2); however, the masses of the fields in a same supermultiplet are related. These relations are precisely the ones which can be found with the method above explained.

I do not review here the explicit calculations [22] which give the mass relations, I give only the result:

$$
\begin{array}{rcll}
m_h^2 = & m_\chi(m_\chi+12)\,, & & \\
m_A^2 = & m_\chi(m_\chi+4) & \text{if} & m_\chi \geq -8\,, \\
m_A^2 = & m_\chi^2+2m_\chi+192 & \text{if} & m_\chi \leq -8\,, \\
m_W^2 = & m_\chi^2+2m_\chi+192 & \text{if} & m_\chi \geq -8\,, \\
m_W^2 = & m_\chi(m_\chi+4) & \text{if} & m_\chi \leq -8\,, \\
m_Z^2 = & (m_\chi+8)(m_\chi+4)\,, & &
\end{array}
\tag{3.1.71}
$$

$$
\begin{array}{rcll}
m_\pi^2 = & m_{\lambda_T}(m_{\lambda_T}+4) & , & \\
m_\phi^2 = & m_{\lambda_T}(m_{\lambda_T}-4) & , & \\
m_A^2 = & m_{\lambda_T}^2-20\,m_{\lambda_T}+96 & \text{if} & m_{\lambda_T} \geq 4\,, \\
m_A^2 = & m_{\lambda_T}(m_{\lambda_T}+4) & \text{if} & m_{\lambda_T} < 4\,, \\
m_W^2 = & m_{\lambda_T}(m_{\lambda_T}+4) & \text{if} & m_{\lambda_T} \geq 4\,, \\
m_W^2 = & m_{\lambda_T}^2-20\,m_{\lambda_T}+96 & \text{if} & m_{\lambda_T} < 4\,, \\
m_Z^2 = & m_{\lambda_T}(m_{\lambda_T}-4)\,, & &
\end{array}
\tag{3.1.72}
$$

$$
\begin{array}{rcll}
m_\pi^2 = & m_{\lambda_L}(m_{\lambda_L}+4) & , & \\
m_S^2 = & (m_{\lambda_L}+24)\,(m_{\lambda_L}+20) & \text{if} & m_{\lambda_L} < -10\,, \\
m_S^2 = & m_{\lambda_L}(m_{\lambda_L}-4) & \text{if} & m_{\lambda_L} \geq -10\,, \\
m_\Sigma^2 = & m_{\lambda_L}(m_{\lambda_L}-4) & \text{if} & m_{\lambda_L} < -10\,, \\
m_\Sigma^2 = & (m_{\lambda_L}+24)\,(m_{\lambda_L}+20) & \text{if} & m_{\lambda_L} \geq -10\,, \\
m_A^2 = & m_{\lambda_L}^2-2\,m_{\lambda_L}+192 & \text{if} & m_{\lambda_L} < -8\,, \\
m_A^2 = & m_{\lambda_L}(m_{\lambda_L}+4) & \text{if} & m_{\lambda_L} \geq -8\,, \\
m_W^2 = & m_{\lambda_L}(m_{\lambda_L}+4) & \text{if} & m_{\lambda_L} < -8\,, \\
m_W^2 = & m_{\lambda_L}^2-2\,m_{\lambda_L}+192 & \text{if} & m_{\lambda_L} \geq -8\,.
\end{array}
\tag{3.1.73}
$$

These supersymmetry relations are pictorially represented in Figure 3.1.

3.2 The M^{111}, Q^{111} and N^{010} spaces

Here and afterwards we set

$$\kappa = 1, \quad e = 1 \tag{3.2.1}$$

which means $R_{AdS_4} = 1/4$, in order to have dimensionless quantities.

3.2.1 M^{111}

Definitions

The M^{pqr} spaces are seven dimensional coset manifolds

$$M^{pqr} = \frac{G}{H} = \frac{SU\,(3) \times SU\,(2) \times U\,(1)}{SU\,(2) \times U\,(1) \times U\,(1)}, \tag{3.2.2}$$

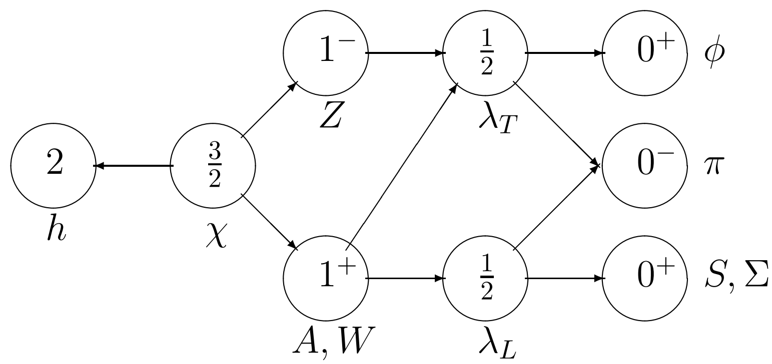


Figure 3.1: Supersymmetry relations between the Kaluza Klein fields: for every couple of fields linked by an arrow there is a mass relation descending by supersymmetry.

where the embedding of H in G is defined as I will explain in the following. They were introduced by E.Witten in the beginning of the eighties [6], with the hope that the four dimensional theory arising from compactification on such a manifold, having as symmetry group $SU(3) \times SU(2) \times U(1)$, could at the end describe standard model physics. Then in [7] the differential geometry of this manifold has been studied, and it has been shown that for every M^{pqr} space an Einstein metric can be defined on it and then there exists a corresponding Freund Rubin solution of eleven dimensional supergravity, and that this solution preserves $\mathcal{N} = 2$ supersymmetry if and only if $p = q$, namely, for the M^{ppr} spaces. Other considerations on these manifolds have been given in [46] and, recently, in [16].

Unfortunately, this was not the right way to obtain the standard model, because chiral fermions cannot arise from these compactifications, and because the anti–de Sitter radius would be unphysical [4]. However, these compactifications acquire a new meaning in the context of AdS/CFT correspondence.

The M^{pqr} spaces can be defined as coset manifolds of the form (3.2.2) where $SU(2) \subset H$ is embedded in $SU(3) \subset G$, and this embedding is such that the fundamental representation of $SU(3)$ decomposes under $SU(2) \subset SU(3)$ as

$$\mathbf{3} \longrightarrow \mathbf{2} \oplus \mathbf{1}. \tag{3.2.3}$$

The (3.2.3) defines univocally the embedding of $SU(2)$ in $SU(3)$ (modulo isomorphisms); the embedding of the two $U(1)$ factors is encoded in the three numbers p, q, r. To define exactly how these numbers determine the embedding of $U(1) \times U(1)$, we give an explicit representation of the group G, by the following 6×6 block–diagonal matrices:

$$G \ni g = \left(\begin{array}{c|c|c} SU(3) & 0 & 0 \\ \hline 0 & SU(2) & 0 \\ \hline 0 & 0 & U(1) \end{array}\right), \tag{3.2.4}$$
$$\underbrace{}_{3}\underbrace{}_{2}\underbrace{}_{1}$$

[4] it is related to the coupling constant of the gauge symmetry G by $e \sim l_p/R_{AdS}$, then if G is the standard model group $e \sim 1$ and $R_{AdS} \sim 10^{-33}$cm!

where the diagonal blocks contain the fundamental representations of $SU(3)$, $SU(2)$ and $U(1)$ respectively. The whole set of generators of G is given by:

$$T_\Lambda \equiv \left(\tfrac{1}{2}\mathrm{i}\lambda_1, \ldots, \tfrac{1}{2}\mathrm{i}\lambda_8, \tfrac{1}{2}\mathrm{i}\sigma_1, \ldots, \tfrac{1}{2}i\sigma_3, iY\right), \tag{3.2.5}$$

where λ_i stands for the i-th Gell-Mann matrix (see appendix A) trivially extended to a 6×6 matrix:

$$\lambda_i \longrightarrow \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.6}$$

Similarly σ_m denotes the following extension of the Pauli matrices:

$$\sigma_m \longrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.2.7}$$

and Y is given by [5]:

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2.9}$$

With these conventions, the $SU\,(2) \subset SU\,(3)$ satisfying the (3.2.3) is generated by $\lambda_1, \lambda_2, \lambda_3$. The remaining two $U\,(1)$ factors in H, whose generators we call Z', Z'', are linear combinations of the three $U\,(1)$ factors in G orthogonal to $SU\,(2)$:

$$\lambda_8,\ \sigma_3,\ Y. \tag{3.2.10}$$

What is relevant is the space generated by Z', Z'', not Z', Z'' themselves; this space is defined giving the combination of the three generators (3.2.10) orthogonal to Z', Z'':

$$Z \equiv p\mathrm{i}\frac{\sqrt{3}}{2}\lambda_8 + q\mathrm{i}\frac{1}{2}\sigma_3 + r\mathrm{i}Y. \tag{3.2.11}$$

Then, a basis for the two abelian generators of H is given by

$$Z' = \sqrt{3}i\lambda_8 + i\sigma_3 - 4iY\,, \tag{3.2.12}$$

$$Z'' = -\tfrac{\sqrt{3}}{2}i\lambda_8 + \tfrac{3}{2}i\sigma_3\,, \tag{3.2.13}$$

which are orthogonal among themselves and with Z: [6]

$$Tr(ZZ') = Tr(ZZ'') = Tr(Z'Z'') = 0\,. \tag{3.2.14}$$

Summarizing, the orthogonal decomposition of the algebra $\mathbb{G}$

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K}, \tag{3.2.15}$$

[5]The normalizations of these generators are chosen to follow the literature [7], [37], [14]. They are normalized so that

$$\mathrm{Tr}\,(T_\Lambda T_{\Lambda'}) = -\frac{1}{2}\delta_{\Lambda\Lambda'}, \tag{3.2.8}$$

with the exception $\mathrm{Tr}\,(YY) = -1$. They are all orthogonal.

[6]But have different norms: for example, when $p = q = r = 1$ $\mathrm{Tr}\,(ZZ) = -3$, $\mathrm{Tr}\,(Z'Z') = -24$, $\mathrm{Tr}\,(Z''Z'') = -6$.

is given by:

$$
\begin{array}{rcl}
 & G & \\
SU(3) & : & \lambda_1, \ldots, \lambda_8 \\
SU(2) & : & \sigma_1, \sigma_2, \sigma_3 \\
U(1) & : & Y \\
 & H & \\
SU(2) & : & \lambda_{\dot{m}} \quad \dot{m} = 1,2,3 \\
U(1) & : & Z' \\
U(1) & : & Z'' \\
 & K & \\
\lambda_A & & A = 4,5,6,7 \\
\sigma_m & & m = 1,2 \\
Z & &
\end{array}
\tag{3.2.16}
$$

where

$$Z \equiv p\mathrm{i}\frac{\sqrt{3}}{2}\lambda_8 + q\mathrm{i}\frac{1}{2}\sigma_3 + r\mathrm{i}Y, \tag{3.2.17}$$

$$Z', Z'' \perp Z. \tag{3.2.18}$$

In this way, the embedding of H in G depends on the choice of the numbers p, q, r.

The generator $Z \in \mathbb{K}$, with these conventions, is

$$Z = \frac{1}{2}\mathrm{i}\begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ 0 & 0 & -2p & 0 & 0 & 0 \\ 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 0 \\ 0 & 0 & 0 & 0 & 0 & 2r \end{pmatrix}. \tag{3.2.19}$$

In order for Z to be the generator of a compact $U(1)$, p, q, r have to be rational; in fact only in this case the application

$$\phi \in I \subset \mathbb{R} \longrightarrow e^{\mathrm{i}Z\phi} \tag{3.2.20}$$

has a compact image. Since Z is defined up to a multiplicative constant (equivalent to a rescaling of ϕ), we can take p, q, r as integer numbers.

Differential geometry and supersymmetry

An explicit parametrization of the coset G/H is given by the seven coordinates (y^A, y^m, y^3):

$$L(y^A, y^m, y^3) = \exp(\tfrac{1}{2}i\lambda_A y^A)\exp(\tfrac{1}{2}i\sigma_m y^m)\exp(Zy^3)\ . \tag{3.2.21}$$

Actually, it is not important that the parametrization is this one: the harmonic analysis formalism does not depend on the coordinate choice.

From the coset representative we can construct the left-invariant one-forms on G/H as:

$$\Omega(y) = L^{-1}(y)\mathrm{d}L(y) = \Omega^{\Lambda}(y)T_{\Lambda}\,, \tag{3.2.22}$$

which satisfies the Maurer-Cartan equations

$$\mathrm{d}\Omega^{\Lambda} + \tfrac{1}{2}C^{\Lambda}{}_{\Sigma\Pi}\Omega^{\Sigma}\wedge\Omega^{\Pi} = 0 \tag{3.2.23}$$

with the structure constants of G:

$$[T_{\Sigma}, T_{\Pi}] = C^{\Lambda}{}_{\Sigma\Pi}T_{\Lambda}\,. \tag{3.2.24}$$

The one-forms Ω^{Λ} can be separated into a set $\{\Omega^{i}\}$ corresponding to the generators of the subalgebra $\mathbb{H}$ and a set $\{\Omega^{a}\}$ corresponding to the coset generators. These latter can be identified with the $SU(3)\times SU(2)\times U(1)$ invariant seven-vielbein on G/H:

$$\mathcal{B}^{a} \equiv (\mathcal{B}^{A}, \mathcal{B}^{m}, \mathcal{B}^{3}),$$

$$\left\{\begin{array}{lcl} \mathcal{B}^{A} &=& \frac{1}{a}\Omega^{A}, \\ \mathcal{B}^{m} &=& \frac{1}{b}\Omega^{m}, \\ \mathcal{B}^{3} &=& \frac{1}{c}(\sqrt{3}\Omega^{8}+\Omega^{3}+2\Omega^{Y}) = \frac{12}{c}\Omega^{Z}, \end{array}\right. \tag{3.2.25}$$

where the multiplicative coefficients define the more general rescaling preserving the G–isometry. The invariant forms Ω^{i} are:

$$\left\{\begin{array}{lcl} \Omega^{\dot{m}}, && \\ \Omega^{Z'} &=& \frac{1}{24}(\sqrt{3}\Omega^{8}+\Omega^{3}-4\Omega^{Y}), \\ \Omega^{Z''} &=& \frac{1}{12}(3\Omega^{3}-\sqrt{3}\Omega^{8}). \end{array}\right. \tag{3.2.26}$$

The spin-connection $\mathcal{B}^{a}{}_{b}$ is determined from the vielbein $\mathcal{B}^{a}$ by imposing vanishing torsion:

$$\mathrm{d}\mathcal{B}^{a} - \mathcal{B}^{a}{}_{b}\wedge\mathcal{B}^{b} = 0, \tag{3.2.27}$$

$$\left\{\begin{array}{lcl} \mathcal{B}^{mn} &=& \epsilon^{mn}\left(\Omega^{3} - \frac{qb^{2}}{2c}\mathcal{B}^{3}\right), \\ \mathcal{B}^{3m} &=& -\frac{qb^{2}}{2c}\epsilon^{mn}\mathcal{B}_{n}, \\ \mathcal{B}^{mA} &=& 0, \\ \mathcal{B}^{3A} &=& -\frac{\sqrt{3}}{2}\frac{pa^{2}}{c}f^{8AB}\mathcal{B}_{B}, \\ \mathcal{B}^{AB} &=& f^{\dot{m}AB}\Omega_{\dot{m}} + f^{8AB}\Omega_{8} - \frac{\sqrt{3}}{2}\frac{pa^{2}}{c}f^{8AB}\mathcal{B}^{3}. \end{array}\right. \tag{3.2.28}$$

Working out the Ricci tensor one finds [7] that for each value of the parameters (p,q,r) there is one and only one value of the rescalings a,b,c such that

$$\mathcal{R}^{a}{}_{b} = 12\delta^{a}_{b}. \tag{3.2.29}$$

Working out the Weyl holonomy algebra of the M^{pqr} manifolds so rescaled, one finds [7] that if $p\neq q$, $\mathcal{G}_{hol} = SO(7)$ and then $\mathcal{N}=0$; if $p=q$, $\mathcal{G}_{hol} = SU(3)$, then $\mathcal{N}_{MAX}=2$, and substituting the null eigenspinors in the (3.1.39) one finds that actually $\mathcal{N}=2$. So the only supersymmetric M^{pqr} manifolds are the M^{ppr}, and have $\mathcal{N}=2$ supersymmetry.

Notice that, as we have seen, since the spaces M^{ppr} preserve $\mathcal{N}=2$ supersymmetries, their isometry group must have the form $G = G'\times SO(2)$. In fact this is the case, being $G = SU(3)\times SU(2)\times U(1)$, $U(1)\simeq SO(2)$, so

$$G' = SU(3)\times SU(2)\,. \tag{3.2.30}$$

p, r **in** M^{ppr}

Let us understand what implies the choice of p, r in the M^{ppr} manifolds. The simplest of the M^{ppr} manifolds is M^{110}. In this case $Z'' \perp Y$, so we can take $Z' \propto Y$, and the $U(1)$ factor in G decouple

$$M^{110} = \frac{SU(3) \times SU(2)}{SU(2) \times U(1)} = \frac{\frac{SU(3)}{SU(2)} \times SU(2)}{U(1)}. \tag{3.2.31}$$

It can be shown that $SU(3)/SU(2) = S^5$, and $SU(2) = S^3$ (locally), so

$$M^{110} = \frac{S^5 \times S^3}{U(1)}. \tag{3.2.32}$$

The $U(1)$ in the denominator is

$$Z'' = -\frac{\mathrm{i}}{2}\left(\sqrt{3}\lambda_8 - 3\sigma_3\right) = -\frac{\mathrm{i}}{2}\,\mathrm{diag}\,(1, 1, -2, -3, 3, 0)\,, \tag{3.2.33}$$

so the ratio of the periods of the $U(1)$ actions on $SU(3)/SU(2)$ and on $SU(2)$ is $3/2$. This manifold is simply connected (see chapter 4).

If $r \neq 0$, we can define $r' = r/p$, and the manifold is

$$M^{ppr} = M^{11r'} = \frac{M^{110} \times U(1)}{U(1)} \tag{3.2.34}$$

where the $U(1)$ factor in the numerator is Y, and the $U(1)$ factor in the denominator is (throwing away the global p^2 multiplicative factor)

$$Z' = r'\mathrm{i}\left(\sqrt{3}\lambda_8 + \sigma_3\right) - 4\mathrm{i}Y = r'\mathrm{i}\;\mathrm{diag}\,(1, 1, -2, 1, -1, -4/r')\,. \tag{3.2.35}$$

Namely, the manifold M^{ppr} is the product of M^{110} with a new one dimensional manifold generated by Y, all quotiented by an identification relation generated by Z'. So points of M^{110} are identified, but are distinguished by the new coordinate. This yields a manifold which sometimes coincide with M^{110}, but in general is

$$M^{ppr} = \frac{M^{110}}{\mathbb{Z}_l}. \tag{3.2.36}$$

In fact points of M^{110} which differ by integer powers of

$$\mathrm{diag}\left(e^{\mathrm{i}\frac{\pi}{2}r'}, e^{\mathrm{i}\frac{\pi}{2}r'}, e^{-\mathrm{i}\pi r'}, e^{\mathrm{i}\frac{\pi}{2}r'}, e^{-\mathrm{i}\frac{\pi}{2}r'}, 1\right) \tag{3.2.37}$$

are identified. But these points, different in $\frac{SU(3)}{SU(2)} \times SU(2)$, could be the same point of M^{110}, namely, they could be already identified in M^{110} by Z'' (3.2.33). If we can find a value of ϕ such that $\exp(\mathrm{i}\phi Z'')$ is equal to the (3.2.37), then $M^{ppr} = M^{110}$. A trivial calculation shows that it always happens if r' is integer, while if $r' = m/n$ (relative primes) there are n points identified. In conclusion, taking p and r relative primes [7],

$$M^{ppr} = \frac{M^{110}}{\mathbb{Z}_p}. \tag{3.2.38}$$

[7] In [46], [37] part of this discussion has been worked out, without taking into account the identification [16] here discussed.

I will consider the simplest M^{ppr} manifold, namely, the simply connected one; as we have shown, all the manifolds M^{11r} with $r \geq 0$ integer coincide; I call this manifold M^{111}. A more detailed geometrical and topological analysis of the M^{111} space is done in the next chapter; a result of this treatment useful in the interpretation of the harmonic analysis results is the following: the second Betti number of M^{111} is $b_2 = 1$, namely, the manifold admits a family of homotopic non–trivial two–cycles (and a family of non–trivial two–forms).

For M^{111} with Einstein metric, taking $e = 1$, the rescaled vielbein is [8]

$$\mathcal{B}^a \equiv (\mathcal{B}^A, \mathcal{B}^m, \mathcal{B}^3),$$

$$\begin{cases} \mathcal{B}^A &= \frac{\sqrt{3}}{8}\Omega^A, \\ \mathcal{B}^m &= \frac{\sqrt{2}}{8}\Omega^m, \\ \mathcal{B}^3 &= \frac{1}{8}(\sqrt{3}\Omega^8 + \Omega^3 + 2\Omega^Y) = \frac{3}{4}\Omega^Z, \end{cases} \tag{3.2.39}$$

and the spin connection is

$$\begin{cases} \mathcal{B}^{mn} &= \epsilon^{mn}\left(\Omega^3 - 2\mathcal{B}^3\right), \\ \mathcal{B}^{3m} &= -2\epsilon^{mn}\mathcal{B}_n, \\ \mathcal{B}^{mA} &= 0, \\ \mathcal{B}^{3A} &= -\frac{4}{\sqrt{3}} f^{8AB}\mathcal{B}_B, \\ \mathcal{B}^{AB} &= f^{\dot{m}AB}\Omega_{\dot{m}} + f^{8AB}\Omega_8 - \frac{4}{\sqrt{3}} f^{8AB}\mathcal{B}^3. \end{cases} \tag{3.2.40}$$

3.2.2 Q^{111}

The Q^{pqr} spaces, found in the eighties [60], are the following coset manifolds:

$$Q^{pqr} = \frac{G}{H} = \frac{SU(2) \times SU(2) \times SU(2)}{U(1) \times U(1)} \tag{3.2.41}$$

where the embedding of the two $U(1)$ is the following; if we take

$$\sigma_i^{(1)},\ \sigma_i^{(2)},\ \sigma_i^{(3)} \tag{3.2.42}$$

as the generators of the three $SU(2)$ factors in G, the maximal torus $U(1) \times U(1) \times U(1) \subset G$ is generated by

$$\sigma_3^{(1)},\ \sigma_3^{(2)},\ \sigma_3^{(3)}; \tag{3.2.43}$$

the generators of $H = U(1) \times U(1)$, which we call Z', Z'', are the combinations of the generators (3.2.43) orthogonal to

$$Z \equiv \frac{\mathrm{i}}{2}p\sigma_3^{(1)} + \frac{\mathrm{i}}{2}q\sigma_3^{(2)} + \frac{\mathrm{i}}{2}r\sigma_3^{(3)}. \tag{3.2.44}$$

So the embedding of H in G is completely defined by the three numbers p, q, r. In order for Z to be the generator of a compact $U(1)$, p, q, r have to be rational numbers (as in

[8]here the relation $\Omega^Z = \frac{1}{6}\left(\sqrt{3}\Omega^8 + \Omega^3 + 2\Omega^Y\right)$ has been found in the following way:

$$\Omega = \Omega^3 T_3 + \Omega^8 T_8 + \Omega^Y T_Y + \ldots = \Omega^Z Z + \ldots,$$
$$Z = \sqrt{3}T_8 + T_3 + T_Y, \quad \mathrm{Tr}\,(Z\Omega) = -1/2\left(\sqrt{3}\Omega^8 + \Omega^3 + 2\Omega^Y\right) = -3\Omega^Z.$$

the M^{pqr} case), and we can take them integers and relative primes by rescaling Z by a multiplicative constant.

By studying the rescaling of the invariant vielbein, one finds that for each value of p, q, r there is one and only one rescaling such that

$$\mathcal{R}^a{}_b = 12\delta^a_b. \tag{3.2.45}$$

By studying the Weyl holonomy, one finds that if $(p, q, r) \neq (p, p, p)$ the Weyl holonomy is $SO(7)$ and so $\mathcal{N} = 0$. If on the contrary $p = q = r$, $\mathcal{G}_{hol} = SU(3)$, so $\mathcal{N}_{MAX} = 2$, and substituting in the (3.1.39) one finds that actually in this case $\mathcal{N} = 2$. I will consider then the manifold Q^{111}, with the vielbein rescaled such that (3.2.45) is satisfied.

In general, the isometry of a coset manifold G/H is not G, but is

$$G \times \left(\frac{N(H)}{H}\right) / U(1)^l \tag{3.2.46}$$

where $N(H)$ is the normalizer of H in G, and $U(1)^l$ are the explicit $U(1)$ factors common in G and $N(H)/H$. This because the generators of $N(H)$ not present in H generate transformations whose *right action* leave invariant the G–left invariant metric; the explicit $U(1)$ factors commute with G, then their right action coincide with their left action, and they are not new symmetries. In the case of M^{111}, $(N(H)/H)/U(1)^l = \{0\}$, but in the case of Q^{111} it is $U(1)$. This is the reason for the apparent contradiction between the $\mathcal{N} = 2$ supersymmetry of Q^{111} and the lack of an explicit $SO(2)$ factor in $G = SU(2) \times SU(2) \times SU(2)$. However, it is possible to describe the Q^{111} manifold by taking into account the normalizer from the start, in a form that exhibits explicitly the complete isometry in G:

$$Q^{111} = \frac{SU(2) \times SU(2) \times SU(2) \times U(1)}{U(1) \times U(1) \times U(1)}, \tag{3.2.47}$$

where the $U(1)^4 \subset G$ is generated by

$$\sigma_3^{(1)},\ \sigma_3^{(2)},\ \sigma_3^{(3)},\ Y, \tag{3.2.48}$$

and H is generated by Z', Z'', Z''' orthogonal to

$$Z = -\frac{\mathrm{i}}{2\sqrt{3}}\left(\sigma_3^{(1)} + \sigma_3^{(2)} + \sigma_3^{(3)}\right) + \frac{\mathrm{i}}{\sqrt{3}}Y. \tag{3.2.49}$$

This form, the one with the $SO(2)$ R–symmetry manifest, is the one which is convenient to use in harmonic analysis, because, as I explain in the next section, only in this way we get a spectrum of fields with well defined R–charge, ready to be organized in supermultiplets.

3.2.3 N^{010}

The N^{pqr} spaces, found in the eighties [46] (see also [59]), are the following coset manifolds:

$$N^{pqr} = \frac{G}{H} = \frac{SU(3) \times U(1)}{U(1) \times U(1)} \tag{3.2.50}$$

where the embedding of the two $U(1)$ is the following; if we take the Gell Mann matrices (see appendix A) as the generators of $SU(3)$ and call Y the generator of the additional $U(1)$ factor in G, the generators of $H = U(1) \times U(1)$ are

$$\begin{aligned} M &= -\frac{\sqrt{2}}{RQ}\left(\tfrac{i}{2}rp\sqrt{3}\lambda_8 + \tfrac{i}{2}rq\lambda_3 - \tfrac{i}{2}(3p^2+q^2)Y\right)\,, \\ N &= -\frac{1}{Q}\left(-\tfrac{i}{2}q\lambda_8 + \tfrac{i}{2}p\sqrt{3}\lambda_3\right)\,, \end{aligned} \tag{3.2.51}$$

with

$$R = \sqrt{3p^2+q^2+2r^2}\,, \qquad Q = \sqrt{3p^2+q^2}\,. \tag{3.2.52}$$

Z, M, N are orthonormalized to $-1/2$. So the embedding of H in G is completely defined by the three numbers p, q, r. In order for Z to be the generator of a compact $U(1)$, p, q, r have to be rational, and as usual we can take them integers relative primes. As for the M^{pqr} spaces, the local geometry depends only from the ratio $x = 3p/q$, while its multiple connectivity depends on r.

One can find that for each p, q, r there are two different rescalings of the invariant vielbein such that

$$\mathcal{R}^a{}_b = 12\delta^a_b, \tag{3.2.53}$$

coincident only if $x = 1$. We can call the corresponding Einstein manifolds N^{pqr}_I and N^{pqr}_{II}. Studying the holonomy and the Killing spinor equation, one finds that:

- the Weyl holonomy of the N^{pqr}_I spaces with $p \neq 0$ is G_2; they have then $\mathcal{N}_{MAX} = 1$, and actually they have $\mathcal{N} = 1$;
- the Weyl holonomy of the N^{0qr}_I spaces is $SU(2)$, so they have $\mathcal{N}_{MAX} = 4$; nevertheless, not all of the solutions of the integrability condition (3.1.49) are actually Killing spinors, namely, solutions of the (3.1.39): the space N^{010}_I admits $\mathcal{N} = 3$ Killing spinors, the other N^{0qr}_I spaces admit $\mathcal{N} = 1$ Killing spinor;
- the Weyl holonomy of the N^{pqr}_{II} spaces is G_2, so $\mathcal{N}_{MAX} = 1$, and they have all $\mathcal{N} = 1$.

In the following, I will consider among these only the space N^{010}_I (and omit the subscript I), which is the only Freund Rubin compactification admitting $\mathcal{N} = 3$ supersymmetries.

In the manifold N^{010} the generators are

$$\begin{aligned} Z &= -\frac{\mathrm{i}}{2}\lambda_3 \\ M &= \frac{\mathrm{i}}{\sqrt{2}}Y \\ N &= \frac{\mathrm{i}}{2}\lambda_8 \end{aligned} \tag{3.2.54}$$

so the $U(1)$ decouples and we have

$$N^{010} = \frac{SU(3)}{U(1)} \tag{3.2.55}$$

where the $U(1)$ generator is $\frac{\mathrm{i}}{2}\lambda_8$. The normalizer of λ_8 in $SU(3)$ is $SU(2)$ (generated by $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$), so the isometry of this manifold is

$$SU(3) \times SU(2) \tag{3.2.56}$$

as foreseen by the fact that it has $\mathcal{N} = 3$ supersymmetry, and $SO(3) \simeq SU(2)$. The G' group is then $SU(3)$.

It has been shown in [46] that this manifold can be realized in a way that makes manifest the all $SU(3) \times SU(2)$ isometry, as

$$N^{010} = \frac{SU(3) \times SU(2)}{SU(2) \times U(1)} \tag{3.2.57}$$

where the $U(1) \subset H$ is generated by

$$T_8^H = \frac{\mathrm{i}}{2}\lambda_8, \tag{3.2.58}$$

and the $SU(2) \subset H$ is *diagonally embedded* into the two $SU(2)$ in G, namely, taking the Pauli matrices as generators of the $SU(2)$ factor in G,

$$T_i^H = \frac{\mathrm{i}}{2}(\lambda_i + \sigma_i) \quad i = 1, 2, 3. \tag{3.2.59}$$

We call this $SU(2) \subset H \quad SU(2)^{diag}$. The generators of the subspace $\mathbb{K}$ in the orthogonal decompositions $\mathbb{G} = \mathbb{H} \oplus \mathbb{K}$ are

$$T_a = \frac{\mathrm{i}}{2}(\lambda_1 - \sigma_1, \lambda_2 - \sigma_2, \lambda_3 - \sigma_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7). \tag{3.2.60}$$

This form, the one with the $SO(\mathcal{N})$ R–symmetry manifest, is the one appropriate for the harmonic analysis.

3.3 Harmonic analysis and mass spectra of Freund Rubin supergravities

Here I review the general theory of harmonic analysis on coset spaces, and its application for the derivation of mass spectra of Freund Rubin G/H supergravities. For a more detailed treatment of this subject see [19], [21], [22], [8], [37].

3.3.1 Harmonics on coset spaces

Let us consider as a first step a group manifold G. A complete functional basis on G is given by the matrix elements of the G UIRs: any function

$$\Phi(g) \quad g \in G \tag{3.3.1}$$

can be expanded as

$$\Phi(g) = \sum_{(\mu)} \sum_{m,n=1}^{\dim(\mu)} c_{mn}^{(\mu)} D_{mn}^{(\mu)}(g) \tag{3.3.2}$$

where (μ) are the UIRs of G, m, n run in these representations, and D are the elements of these representations. In fact, the $D^{(\mu)}_{mn}(g)$ satisfy the orthogonality and completeness relations (see [61], p.172)

$$\begin{aligned}
&\int_G dg D^{(\mu)}_{mn}(g)\, D^{(\nu)}_{sr}\left(g^{-1}\right) = \frac{\mathrm{vol}\,(G)}{\mathrm{vol}\,(\mu)}\delta_{mr}\delta_{ns}\delta^{(\mu)(\nu)} \\
&\sum_{(\mu)} D^{(\mu)}_{mn}(g) D^{(\mu)}_{nm}(g'^{-1})\mathrm{dim}\,(\mu) = \delta\,(g-g')\,\mathrm{vol}\,(G)\,.
\end{aligned} \tag{3.3.3}$$

If $\Phi\,(g)$ transforms in an irreducible representation (μ) of G, for example under left multiplication, namely

$$\Phi^{(\mu)}_m\,(g'g) = D^{(\mu)}_{mn}\,(g')\,\Phi^{(\mu)}_n\,(g)\,, \tag{3.3.4}$$

then only a subset of the complete functional basis is present, the D's that transform in the same way, that is,

$$D^{(\mu)}_{mn} \qquad \mu, m \text{ fixed.} \tag{3.3.5}$$

So in this case the expansion is shorter:

$$\Phi^{(\mu)}_m\,(g) = \sum_n c^{(\mu)}_n D^{(\mu)}_{mn}\,(g)\,. \tag{3.3.6}$$

Let us now consider the functions $\Phi\,(y)$ on a coset manifold G/H. The matrix elements

$$D^{(\mu)}_{mn}\,(L\,(y))\,,$$

with $L\,(y)$ coset representative of the coset space, y coset coordinate, are a complete functional basis on G/H:

$$\Phi\,(L\,(y)) = \sum_{(\mu)} \sum_{m,n=1}^{\mathrm{dim}(\mu)} c^{(\mu)}_{mn} D^{(\mu)}_{mn}\,(L\,(y)) \tag{3.3.7}$$

satisfying

$$\begin{aligned}
&\int_{G/H} d\mu(y)\, D^{(\mu)}_{mn}\,(g)\, D^{(\nu)}_{sr}\left(g^{-1}\right) = \frac{\mathrm{vol}\,(G/H)}{\mathrm{vol}\,(\mu)}\delta_{mr}\delta_{ns}\delta^{(\mu)(\nu)} \\
&\sum_{(\mu)} D^{(\mu)}_{mn}\,(g)\, D^{(\mu)}_{nm}(g'^{-1})\mathrm{dim}\,(\mu) = \delta\,(g-g')\,\mathrm{vol}\,(G/H)
\end{aligned} \tag{3.3.8}$$

where $d\mu(y)$ is the invariant measure on G/H.

We are interested on functions $\Phi\,(L\,(y))$ on which an action of H is defined, that transform in an irreducible representation (ρ) of H

$$h \cdot \Phi^{(\rho)}_i\,(L\,(y)) \equiv D_{ij}\,(h)^{(\rho)}\,\Phi^{(\rho)}_j\,(L\,(y)) \tag{3.3.9}$$

where the index i runs in (ρ). Which are the functions among the

$$D^{(\mu)}_{mn} \quad m, n \text{ running in } (\mu) \text{ of } G \tag{3.3.10}$$

that transform in this way? They are the functions

$$D^{(\mu)}_{in} \quad i \text{ running in } (\rho) \text{ of } H,\; n \text{ running in } (\mu) \text{ of } G \tag{3.3.11}$$

but i has to run also in the (μ) of G, then not all the G representations (μ) are appropriate, only the (μ) that satisfy the following condition: the decomposition of (μ) with respect to $H \subset G$ must contain the H irreducible representation (ρ):

$$(\mu) \xrightarrow{H} \cdots \oplus (\rho) \oplus \cdots . \tag{3.3.12}$$

Only in this case $D^{(\mu)}_{mn}$ decomposes in $\left(\ldots, D^{(\mu)}_{in}, D^{(\mu)}_{i'n}, \ldots\right)$ and the $D^{(\mu)}_{in}$ actually exists. The functions satisfying the (3.3.11), (3.3.12) are called $H-$**harmonics** on G/H, and constitute a complete basis for the coset function $\Phi^{(\rho)}_i(L(y))$. Its expansion is

$$\Phi^{(\rho)}_i(L(y)) = \sum_{(\mu)}{}' \sum_n c^{(\mu)}_n D^{(\mu)}_{in}(L(y)) \tag{3.3.13}$$

where $\sum'$ means a sum only on the representations (μ) satisfying the property (3.3.12). Notice that the $H-$harmonics have both an index running in an irreducible representation of G (on the right) and an index running in an irreducible representation of H (on the left). The coefficients of the expansion $c^{(\mu)}_n$ have an index of a representation of G present in the expansion $\sum'$.

3.3.2 Differential operators on H harmonics

The H–harmonics have a very powerful property: it is possible to express the action of differential operators on them in an algebraic way. In fact, as we has seen in (3.1.34), the action of the H–covariant derivative on the inverse coset representative is

$$\mathcal{D}^H_a L^{-1} = -r_a T_a L^{-1} \quad \text{no sum on } a \tag{3.3.14}$$

with T_a generator of the subspace $\mathbb{K}$ defined by the orthogonal decomposition $\mathbb{G} = \mathbb{K} \oplus \mathbb{H}$, in the representation in which the inverse coset representative is expressed, and r_a rescaling of the vielbein. But the harmonics *are* the inverse coset representatives in the representation (μ):

$$D^{(\mu)}{}^i{}_n = L^{-1}{}^i{}_n . \tag{3.3.15}$$

More precisely, the harmonic in the (ρ, μ^t), $D^{(\mu)}{}^i{}_n$, is obtained doing the decomposition (3.3.12) of the first index of $D^{(\mu)}{}^m{}_n = L^{-1}{}^m{}_n$ and taking the (ρ) term. We consider the *inverse* coset representative because for simplicity of notation we want H to act on their left, while it acts on the right of coset representatives.

The action of $(T^{(\mu)}_a)^m_n$ on $L^{-1}{}^i{}_m$ is

$$\mathcal{D}^H_a \left(D^i{}_n\right) = -r_a \left(T_a D\right)^i{}_n = -r_a \left(T_a\right)^i_m D^m_n \tag{3.3.16}$$

where $(T_a)^i{}_m$ is defined as the (ρ) term in the decomposition (3.3.12) of the index n in $(T_a)^n{}_m$, namely,

$$(T_a)^n{}_m = \left\{\ldots, (T_a)^i{}_m, \ldots\right\} . \tag{3.3.17}$$

As we have seen, all the operators (3.1.63),. . .,(3.1.69) can be built with the $SO(7)$ covariant derivative and the G/H Killing metric. Furthermore, from the (3.1.27)

$$\mathcal{D}^{SO(7)} = \mathcal{D}^H + \mathbb{M}_a \mathcal{B}^a \tag{3.3.18}$$

we can write the $SO(7)$ covariant derivative in terms of the H covariant derivative. Then, the action of all the operators (3.1.63),. . .,(3.1.69) on the harmonics can be expressed algebraically.

3.3.3 Harmonic expansion of supergravity fields

The fluctuations of eleven dimensional supergravity fields around the Freund-Rubin solution, defined in (3.1.55), are fields on $AdS_4 \times G/H$:

$$\Phi^{[E\,s]}_{[\lambda_1\lambda_2\lambda_3]\hat{a}}(x,y) \tag{3.3.19}$$

where $\hat{a}$ is an index in the $[\lambda_1\lambda_2\lambda_3]$ representation of $SO(7)$. We leave implicit the spacetime index in the $[E\,s]$ of $SO(3,2)$ because we are not interested on it. We know how to expand in harmonics a field lying in an H representation, but Φ is in an $SO(7)$ representation. However $H \subset G$, embedding defined by the (3.1.4):

$$C_{ia}{}^{b} = -\left(T^H_i\right)_a{}^b = \left(T^H_i\right)^b{}_a\,, \tag{3.3.20}$$

then given the generators of $SO(7)$ in a generic representation, $t^{SO(7)}_{ab}$, the generators of H in that representations are

$$T^H_i = C_{ia}{}^{b}\left(t^{SO(7)}\right)^a{}_b\,. \tag{3.3.21}$$

This defines the decomposition of the $SO(7)$ irreducible representations $[\lambda_1,\lambda_2,\lambda_3]$ in H irreducible representations. In this way we can decompose the Φ field in fragments which are in irreducible representations of H

$$\Phi^{[E\,s]}_{[\lambda_1\lambda_2\lambda_3]\hat{a}}(x,y) = \left\{\Phi^{[E\,s]}_{(\rho_1)i_1}(x,y),\ldots,\Phi^{[E\,s]}_{(\rho_r)i_r}(x,y)\right\}. \tag{3.3.22}$$

Each of these fragments can be expanded in H–harmonics

$$\Phi^{[E\,s]}_{(\rho_\xi)i_\xi}(x,y) = \sum_{(\mu)}{}' \sum_{n=1}^{\dim(\mu)} {}_\xi\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_\xi)i_\xi}\left(L\left(y\right)\right) \quad \xi = 1,\ldots,r \tag{3.3.23}$$

where we denote the harmonics $\mathcal{H}_i{}^n \equiv D_i{}^n$. Here each ϕ is one of the AdS_4 fields listed in section 22, and is in a representation of G. So we have

$$\Phi^{[E\,s]}_{[\lambda_1,\lambda_2,\lambda_3]\hat{a}}(x,y) = \sum_{(\mu)}{}' \sum_{n}\left\{{}_1\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_1)i_1}(y),\ldots,{}_r\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_1)i_r}(y)\right\}. \tag{3.3.24}$$

This is the expansion given in the (3.1.61) [9], where it was written in the simpler but less precise form $\Phi^{[E\,s]}_{\hat{a}}(x,y) = \phi^{[E\,s]}_n(x)\,\mathcal{H}^n_{\hat{a}}(y)$.

3.3.4 The mass spectrum from harmonic analysis

In order to find the masses of the AdS_4 supergravity fields we have to solve the eigenvalue equations (3.1.63),...,(3.1.69), which all have the form

$$\boxtimes^{[\lambda_1,\lambda_2,\lambda_3]}_y \mathcal{H}^{(\mu)m}_{[\lambda_1,\lambda_2,\lambda_3]\hat{a}}(y) = M_{[\lambda_1,\lambda_2,\lambda_3]}\mathcal{H}^{(\mu)m}_{[\lambda_1,\lambda_2,\lambda_3]\hat{a}}(y)\,. \tag{3.3.25}$$

[9] with little notation differences: in the (3.1.61) the name of the G representation, (μ), is not explicit, and the index running in this representation is called I instead of n

These eigevalues yield the masses of the fields by the formulae (3.1.70).

Harmonic analysis allows us to find the eigenvalues of the (3.3.25) with purely algebraic calculations, without solving any differential equation, and without requiring an explicit coordinatization of the manifold. To do this, we have to write the (3.3.25), which as we have seen is a shorthand notation but not actually the exact expression, in a precise form. First of all we have to consider the decomposition under H of the given $SO(7)$ representation $[\lambda_1, \lambda_2, \lambda_3]$,

$$[\lambda_1, \lambda_2, \lambda_3] \xrightarrow{H} (\rho_1) \oplus \cdots \oplus (\rho_r) \tag{3.3.26}$$

where r is the number of fragments in this decomposition. We have to keep attention on the fragments which appear more than one time in the decomposition: r is the number of H UIRs times their multiplicity.

Then, we determine which G UIRs satisfy the condition (3.3.12), namely, do contain in their H decompositions the representations present in the (3.3.26). The expansion contains only that G representations (μ). The eigenvalue equation has the form:

$$\boxtimes_y^{[\lambda_1,\lambda_2,\lambda_3]} \sum_{(\mu)}{}' \sum_n \begin{pmatrix} {}_1\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_1)i_1}(y) \\ \vdots \\ {}_r\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_r)i_r}(y) \end{pmatrix} = M_{[\lambda_1,\lambda_2,\lambda_3]} \sum_{(\mu)}{}' \sum_n \begin{pmatrix} {}_1\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_1)i_1}(y) \\ \vdots \\ {}_r\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_r)i_r}(y) \end{pmatrix}. \tag{3.3.27}$$

Now, we determine the action of the H covariant derivative on the fragments $\mathcal{H}^{(\mu)n}_{(\rho_i)i}$ given by the (3.3.16), and by means of the (3.3.18), the action of the $SO(7)$ covariant derivative and then of the invariant operator $\boxtimes_y^{[\lambda_1,\lambda_2,\lambda_3]}$. This action, in general, sends a fragment $\mathcal{H}^{(\mu)n}_{(\rho_\xi)i_\xi}$ in a fragment $\mathcal{H}^{(\mu)n}_{(\rho_{\xi'})i_{\xi'}}$ with $i_{\xi'}$ running in $\rho_{\xi'}$ which is another H representation of the decomposition (3.3.12). However, for an operator $\boxtimes_y^{[\lambda_1,\lambda_2,\lambda_3]}$ among the (3.1.63),. . .,(3.1.69), this new fragment has to be present in the decomposition of $[\lambda_1, \lambda_2, \lambda_3]$. So, if we consider the r dimensional vector space with base vectors

$$e_\xi \equiv \sum_{(\mu)}{}' \sum_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ {}_\xi\phi^{[E\,s]}_{(\mu)n}(x)\,\mathcal{H}^{(\mu)n}_{(\rho_\xi)i_\xi}(y) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.3.28}$$

the invariant operator acts as an $r \times r$ numeric matrix on this vector space

$$\boxtimes_y^{[\lambda_1,\lambda_2,\lambda_3]} e_\xi = \mathcal{M}_\xi{}^{\xi'} e_{\xi'}. \tag{3.3.29}$$

All we have to do is to construct this matrix, whose entries depend on the labels of the G UIR, and to find its eigenvalues $M_{[\lambda_1,\lambda_2,\lambda_3]}$. The corresponding eigenvectors are the supergravity fields. Substituting these eigenvalues in the (3.1.70) we find the masses of the corresponding AdS_4 fields listed in section 22. In this way the complete spectrum of compactified supergravity can be worked out.

It is worth noting that the AdS_4 fields so found are in G representations. I remind that $G = G' \times SO(\mathcal{N})$, and $SO(\mathcal{N})$ is the R–symmetry group of the theory. So in this way we find not only the masses of the fields with various spins (namely, their $SO(3,2)$ UIRs), but also their R–symmetry labels. We can then organize them in supermultiplets of $\mathcal{N}$–extended supersymmetry. This is the reason we prefer coset spaces in a form which exhibits explicitly the complete isometry. All the fields in a same supermultiplet have the same G' labels, so each supermultiplet is in a G' UIR. The final result of the harmonic analysis, then, is the list of all the supermultiplets present in the theory and of their G' representations.

Furthermore, the mass relations (3.1.71), (3.1.72), (3.1.73) give the mass of every field in a supermultiplet in terms of the mass of some other field in the same supermultiplet. This is a very strong check against errors, because from the analysis of an $SO(7)$ harmonics we know what we expect from other $SO(7)$ harmonics. And this allows us to skip the more complicate operators: thanks of the constraint of supersymmetry - namely, the fields have to make supermultiplets with same G' labels and masses related by the mass relations (3.1.71), (3.1.72), (3.1.73) - only the harmonic analysis of a part of the operators (3.1.63),. . ., (3.1.69) is necessary. Finally, if when we start the harmonic analysis we do not know completely the structure of the multiplets, we can use the harmonic analysis itself and the mass relations to fill the blanks in our knowledge. This is what we have actually done: the derivation of the spectrum of $AdS_4 \times M^{111}$ supergravity (see next section and [14]) allowed us to complete the structure of $\mathcal{N} = 2$ supermultiplets, the derivation of the spectrum of $AdS_4 \times N^{010}$ supergravity [53], [17] allowed us to complete the structure of $\mathcal{N} = 3$ supermultiplets, yielding the tables given in chapter 2.

3.4 The mass spectrum of $AdS_4 \times M^{111}$ supergravity

This section is based on the work done in the collaboration [14]. However, part of the spectrum had been worked out previously [21], [8]. I start giving the result, then I explain how it has been found. The conventions relative to M^{111} space are given in appendix A.

3.4.1 Representations of G and H

Here we fix the conventions for labelling the irreducible representations of

$$G' = SU(3) \times SU(2)\,. \tag{3.4.1}$$

It has rank three, so that its irreducible representations are labeled by three integer numbers. A representation of $SU(3)$ can be identified by a Young diagram of the following type

$$\underbrace{\begin{array}{|c|c|c|}\hline \;\; & \cdots & \;\; \\ \hline \;\; & \cdots & \;\; \\ \hline\end{array}}_{M_2}\underbrace{\begin{array}{|c|c|c|}\hline \;\; & \cdots & \;\; \\ \hline \end{array}}_{M_1},$$

while an UIR of $SU(2)$ can be identified by a Young diagram as follows

$$\underbrace{\begin{array}{|c|c|c|}\hline \;\; & \cdots & \;\; \\ \hline\end{array}}_{2J}.$$

Hence we can take the nonnegative integers $M_1,\ M_2,\ 2J$, as the labels of a G' irreducible representation.

An UIR of $G = SU(3) \times SU(2) \times U(1)$ will then be denoted by

$$[M_1.M_2, J, Y] \tag{3.4.2}$$

where Y is the charge of the $U(1)$ factor in G, namely, the hypercharge. An UIR of $H = SU(2) \times U(1) \times U(1)$ will be denoted by

$$\left[J^h, Z', Z''\right] \tag{3.4.3}$$

where Z', Z'' are the charges of the $U(1)$'s generated by Z' and Z'', and J^h is the $SU(2)$ spin defined as above; the superscript c distinguishes it from the label J of the $SU(2) \subset G$ representation.

3.4.2 Results

Relying on the procedures explained in the following sections, we have found the following results.

Not every G' representation is actually present, but only those representations that satisfy the following relations

$$M_2 - M_1 \in 3\,\mathbb{Z} \quad ; \quad J \in \mathbb{N}. \tag{3.4.4}$$

In the following pages, for each type of $\mathcal{N} = 2$ multiplet I list the G' representations through which it occurs in the spectrum. I do this by writing bounds on the range of values for the $M_1,\ M_2,\ 2J$ labels. The reader should take into account that, case by case, in addition to the specific bounds written, also the general restriction (3.4.4) has to be imposed.

Furthermore for every multiplet, I give the energy and hypercharge values E_0 and y_0 of the Clifford vacuum. From the tables 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15 it is straightforward to get the energies and the hypercharges of all other fields in each multiplet.

As a short–hand notation let us name H_0 the following quadratic form in the representation labels:

$$H_0 \equiv \frac{64}{3}\left(M_2 + M_1 + M_2 M_1\right) + 32J\left(J+1\right) + \frac{32}{9}\left(M_2 - M_1\right)^2. \tag{3.4.5}$$

Up to multiplicative constants, the first two addenda $M_2 + M_1 + M_2\,M_1$ and $J(J+1)$ are the Casimirs of $G' = SU(3) \times SU(2)$. The last addendum is contributed by the square of the hypercharge through its relation with the $SU(3)$ representation implied by the geometry of the space.

I remind that when $y_0 = 0$ the multiplet is real, while when $y_0 \neq 0$ it is complex, and the number of the degrees of freedom is doubled; in the latter case, I write only the multiplet with positive hypercharge (or, in few cases, negative); the one with negative (positive) hypercharge, and conjugate flavour indices $(M_1, M_2, J) \longrightarrow (M_2, M_1, J)$, is its complex conjugate.

1. **Long graviton multiplets**

$$
\begin{aligned}
\text{complex } (y_0 \neq 0) \quad &: \quad \left(2\,(2)\,, 8\left(\frac{3}{2}\right), 12\,(1)\,, 8\left(\frac{1}{2}\right), 2\,(0)\right) \\
\text{real } (y_0 = 0) \quad &: \quad \left(1\,(2)\,, 4\left(\frac{3}{2}\right), 6\,(1)\,, 4\left(\frac{1}{2}\right), 1\,(0)\right)
\end{aligned}
$$

One long graviton multiplet (table 3.7) in each representation of the series

$$
\begin{aligned}
&\left\{M_2 \geq M_1 \geq 0,\ J > \frac{1}{3}\,(M_2 - M_1)\right\} \cup \\
&\left\{M_2 \geq M_1 > 0,\ J = \frac{1}{3}\,(M_2 - M_1)\right\}
\end{aligned}
\tag{3.4.6}
$$

with

$$
h:\ E_0 = \frac{1}{2} + \frac{1}{4}\sqrt{H_0 + 36},\ y_0 = \frac{2}{3}\,(M_2 - M_1) \tag{3.4.7}
$$

2. **Long gravitino multiplets**

$$
\text{complex } (y_0 \neq 0) \quad : \quad \left(2\left(\frac{3}{2}\right), 8\,(1)\,, 12\left(\frac{1}{2}\right), 8\,(0)\right)
$$

There are two different realizations of the long gravitino multiplet, χ^+ with positive mass and χ^- with negative mass.

- Four long gravitino multiplets (two χ^+ and two χ^-, table 3.8) in each representation of the series

$$
\begin{aligned}
&\left\{M_2 \geq M_1 > 0,\ J > \frac{1}{3}\,(M_2 - M_1) + 1\right\} \cup \\
&\left\{M_2 \geq M_1 > 1,\ J = \frac{1}{3}\,(M_2 - M_1) + 1\right\}
\end{aligned}
\tag{3.4.8}
$$

  with

$$
\begin{aligned}
\chi^+ :&\quad E_0 = -\tfrac{1}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}\,(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}\,(M_2 - M_1) - 1 \\
\chi^+ :&\quad E_0 = -\tfrac{1}{2} + \tfrac{1}{4}\sqrt{H_0 - \tfrac{32}{3}\,(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}\,(M_2 - M_1) + 1 \\
\chi^- :&\quad E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}\,(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}\,(M_2 - M_1) - 1 \\
\chi^- :&\quad E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{H_0 - \tfrac{32}{3}\,(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}\,(M_2 - M_1) + 1\,.
\end{aligned}
\tag{3.4.9}
$$

- Three long gravitino multiplets (one χ^+ and two χ^-, table 3.8), in each representation of the series

$$\left\{M_2 \geq M_1 = 1,\ J = \frac{1}{3}(M_2 - M_1) + 1\right\} \tag{3.4.10}$$

with

$$\begin{aligned}
\chi^+ : \quad & E_0 = -\tfrac{1}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}(M_2 - M_1) - 1 \\
\chi^- : \quad & E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{H_0 - \tfrac{32}{3}(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}(M_2 - M_1) + 1 \\
\chi^- : \quad & E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}(M_2 - M_1) - 1\,.
\end{aligned} \tag{3.4.11}$$

- Two long gravitino multiplets (one χ^+ and one χ^-, table 3.8) in each representation of the series

$$\begin{gathered}
\left\{M_2 > M_1 > 0,\ J = \frac{1}{3}(M_2 - M_1)\right\} \cup \\
\left\{M_2 > M_1 > 0,\ J = \frac{1}{3}(M_2 - M_1) - 1\right\} \cup \\
\left\{M_2 > M_1 = 0,\ J \geq \frac{1}{3}(M_2 - M_1)\right\}
\end{gathered} \tag{3.4.12}$$

with

$$\begin{aligned}
\chi^+ : \quad & E_0 = -\tfrac{1}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}(M_2 - M_1) - 1 \\
\chi^- : \quad & E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}, \quad y_0 = \frac{2}{3}(M_2 - M_1) - 1\,.
\end{aligned} \tag{3.4.13}$$

- One long gravitino multiplet (a χ^-, table 3.8), in each representation of the series

$$\left\{M_2 > M_1 = 0,\ J = \frac{1}{3}(M_2 - M_1) - 1\right\} \tag{3.4.14}$$

with

$$\chi^- :\ E_0 = \frac{3}{2} + \frac{1}{4}\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16},\ y_0 = \frac{2}{3}(M_2 - M_1) - 1\,. \tag{3.4.15}$$

3. **Long vector multiplets**

$$\begin{aligned}
\text{complex } (y_0 \neq 0) \quad &: \quad \left(2\,(1)\,, 8\left(\frac{1}{2}\right), 10\,(0)\right) \\
\text{real } (y_0 = 0) \quad &: \quad \left(1\,(1)\,, 4\left(\frac{1}{2}\right), 5\,(0)\right)
\end{aligned}$$

As already stressed there are different realizations of the long vector multiplet arising from different fields of the $D = 11$ theory. We have the W vector multiplets, the A vector multiplets, and the Z vector multiplets.

- One W long vector multiplet (table 3.9) in each representation of the series

$$\left\{M_2 \geq M_1 \geq 0,\ J \geq \frac{1}{3}(M_2 - M_1)\right\} \tag{3.4.16}$$

with

$$W:\ E_0 = \frac{5}{2} + \frac{1}{4}\sqrt{H_0 + 36},\ y_0 = \frac{2}{3}(M_2 - M_1)\ . \tag{3.4.17}$$

- One A long vector multiplet (table 3.9) in each representation of the series

$$\begin{gathered}\left\{M_2 \geq M_1 = 0,\ J > \frac{1}{3}(M_2 - M_1) + 1\right\} \cup \\ \left\{M_2 \geq M_1 = 1,\ J > \frac{1}{3}(M_2 - M_1)\right\} \cup \\ \left\{M_2 \geq M_1 > 1,\ J \geq \frac{1}{3}(M_2 - M_1)\right\}\end{gathered} \tag{3.4.18}$$

with

$$A:\ E_0 = -\frac{3}{2} + \frac{1}{4}\sqrt{H_0 + 36},\ y_0 = \frac{2}{3}(M_2 - M_1)\ . \tag{3.4.19}$$

- One Z long vector multiplet (table 3.9) in each representation of the series

$$\left\{M_2 \geq M_1 > 0,\ J \geq \frac{1}{3}(M_2 - M_1)\right\} \tag{3.4.20}$$

with

$$Z:\ E_0 = \frac{1}{2} + \frac{1}{4}\sqrt{H_0 + 4},\ y_0 = \frac{2}{3}(M_2 - M_1)\ . \tag{3.4.21}$$

- One Z long vector multiplet (table 3.9) in each representation of the series

$$\begin{gathered}\left\{M_2 > M_1 + 3,\ J \geq \frac{1}{3}(M_2 - M_1) - 2\right\} \cup \\ \left\{M_1 + 3 \geq M_2 > 1,\ J > -\frac{1}{3}(M_2 - M_1) + 1\right\}\end{gathered} \tag{3.4.22}$$

with

$$Z:\ E_0 = \frac{1}{2} + \frac{1}{4}\sqrt{H_0 + \frac{64}{3}(M_2 - M_1) - 28},\ y_0 = \frac{2}{3}(M_2 - M_1) - 2\,. \tag{3.4.23}$$

SHORT MULTIPLETS

They are always complex.

1. <u>**Short graviton multiplets**</u> $\left(2\,(2)\,,6\left(\frac{3}{2}\right),6\,(1)\,,2\left(\frac{1}{2}\right)\right)$

- One short graviton multiplet (table 3.10) in each representation of the series

$$\begin{cases} M_2 = 3k \\ M_1 = 0 \\ J = k \end{cases} \quad k > 0 \text{ integer} \tag{3.4.24}$$

with

$$E_0 = 2k + 2, \; y_0 = 2k\,. \tag{3.4.25}$$

2. **Short gravitino multiplets** $\left(2\left(\frac{3}{2}\right), 6\,(1), 6\left(\frac{1}{2}\right), 2\,(0)\right)$

- One short gravitino multiplet (χ^+, table 3.11) in each representation of the series

$$\begin{cases} M_2 = 3k+1 \\ M_1 = 1 \\ J = k+1 \end{cases} \quad k \geq 0 \text{ integer} \tag{3.4.26}$$

with

$$E_0 = 2k + \frac{5}{2}, \; y_0 = 2k+1\,. \tag{3.4.27}$$

- One short gravitino multiplet (χ^+, table 3.11) in each representation of the series

$$\begin{cases} M_2 = 3k+3 \\ M_1 = 0 \\ J = k \end{cases} \quad k \geq 0 \text{ integer} \tag{3.4.28}$$

with

$$E_0 = 2k + \frac{5}{2}, \; y_0 = 2k+1\,. \tag{3.4.29}$$

3. **Short vector multiplets** $\left(2\,(1), 6\left(\frac{1}{2}\right), 6\,(0)\right)$

- One short vector multiplet (A, table 3.12), in each representation of the series

$$\begin{cases} M_2 = 3k+1 \\ M_1 = 1 \\ J = k \end{cases} \quad k > 0 \text{ integer} \tag{3.4.30}$$

$$\begin{cases} M_2 = 3k \\ M_1 = 0 \\ J = k+1 \end{cases} \quad k > 0 \text{ integer} \tag{3.4.31}$$

with

$$E_0 = 2k + 1, \; y_0 = 2k\,. \tag{3.4.32}$$

4. **Hypermultiplets** $\left(2\left(\frac{1}{2}\right), 4\,(0)\right)$

- One hypermultiplet (table 3.13) in each representation of the series

$$\begin{cases} M_2 = 3k \\ M_1 = 0 \\ J = k \end{cases} \quad k > 0 \text{ integer} \tag{3.4.33}$$

$$E_0 = |y_0| = 2k\,. \tag{3.4.34}$$

They are always real.

1. **The massless graviton multiplet** (table 3.14) $\left(1\,(2)\,,2\left(\frac{3}{2}\right),1\,(1)\right)$
in the singlet representation

$$M_2 = M_1 = J = 0 \tag{3.4.35}$$

with

$$E_0 = 2,\ y_0 = 0. \tag{3.4.36}$$

In this multiplet the graviphoton is associated with the Killing vector of the R–symmetry group $U\,(1)_R$.

2. **The massless vector multiplet** (table 3.15) $\left(1\,(1)\,,2\left(\frac{1}{2}\right),2\,(0)\right)$
in the *adjoint representation* of the G' group

$$M_2 = M_1 = 1,\ J = 0 \tag{3.4.37}$$

$$M_2 = M_1 = 0,\ J = 1 \tag{3.4.38}$$

with

$$E_0 = 1,\ y_0 = 0. \tag{3.4.39}$$

3. **An additional massless vector multiplet** in the singlet representation of the gauge group

$$M_2 = M_1 = J = 0 \tag{3.4.40}$$

with the same energy and hypercharges as in (3.4.39) that arises from the three–form a_{mab} and is due to the existence of one closed cohomology two–form on the M^{111} manifold. This multiplet is named the Betti multiplet.

Summarizing, the massless spectrum, besides the supergravity multiplet contains twelve vector multiplets: so the total number of massless gauge bosons is thirteen, one of them being the graviphoton. In the low energy effective lagrangian we just couple to supergravity these twelve vector multiplets. However we expect the gauging of a thirteen–parameter group:

$$SU\,(3) \times SU\,(2) \times U(1)_R \times U\,(1)' \tag{3.4.41}$$

the further $U\,(1)'$ being associated with the Betti multiplet. All Kaluza Klein states are neutral under $U\,(1)'$ yet non perturbative states can carry $U(1)'$ charges. This actually happens, as I will show in chapter 4.

3.4.3 Harmonic expansion on M^{111}

In terms of the structure constant of G, given in appendix A, we find that the embedding of the algebra $\mathbb{H}$ into the adjoint representation of $SO(7)$ is

$$(T_H)^a{}_b = C_{Hb}{}^a,$$

$$(T_{Z'})^a{}_b = \begin{pmatrix} 2\sqrt{3}f^{8AB} & 0 & 0 \\ 0 & 2\epsilon^{mn} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4.42}$$

$$(T_{Z''})^a{}_b = \begin{pmatrix} -\sqrt{3}f^{8AB} & 0 & 0 \\ 0 & 3\epsilon^{mn} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.4.43}$$

$$(T_{\dot m})^a{}_b = \begin{pmatrix} f^{\dot m AB} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.4.44}$$

This means that the $SO(7)$-indices of the various n-forms can be split in the following subsets, each one transforming into an irreducible representation of H:

$$\begin{array}{rcl} \mathcal{Y}^a & = & \{\mathcal{Y}^A, \mathcal{Y}^m, \mathcal{Y}^3\} \\ \mathcal{Y}^{[ab]} & = & \{\mathcal{Y}^{AB}, \mathcal{Y}^{Am}, \mathcal{Y}^{mn}, \mathcal{Y}^{A3}, \mathcal{Y}^{m3}\} \\ \mathcal{Y}^{[abc]} & = & \{\mathcal{Y}^{ABC}, \mathcal{Y}^{ABm}, \mathcal{Y}^{AB3}, \mathcal{Y}^{Amn}, \mathcal{Y}^{Am3}, \mathcal{Y}^{mn3}\} \end{array} \tag{3.4.45}$$

and the $SO(7)$ irreducible representations $[\lambda_1\lambda_2\lambda_3]$ break into the direct sum of H irreducible representations. The $[J^h, Z, Z']$ labels of every H–irreducible fragment can be read off from the action of $T_{\dot m}$, T_Z, $T_{Z'}$ on that representation.

The expansion (3.3.23) of a generic $SO(3,2)\times H$-irreducible field is

$$\Phi^{[E\,s]}_{[J^h Z' Z'']i_1\cdots i_{2J^h}}(x,y) = {\sum}'_{[M_1M_2JY]} \sum_{\zeta} \sum_{m} \mathcal{H}^{[M_1M_2JY]m\zeta}_{[J^h Z' Z'']i_1\cdots i_{2J^h}}(y)\cdot \varphi^{[E\,s]}_{[M_1M_2JY]m\zeta}(x)\,. \tag{3.4.46}$$

The coefficients $\varphi(x)$ of the expansion become the space–time fields of the theory in AdS_4. The first sum is over all the G irreducible representations $[M_1M_2JY]$ which break into the given H-one. We call $\sum'$ the sum over this subset of the possible representations of G. The subscripts ${}_{i_1,\cdots,i_{2J^h}}$ span the representation space of $[J^h Z' Z'']$, while m is a collective index which spans the representation space of $[M_1M_2JY]$. Finally ζ accounts for the fact that the same H irreducible representation can be embedded in G in different ways. In fact, given an $SU(3)$ representation $[M_1,\ M_2]$, it can contain the $SU(2)\subset SU(3)$ representation $[J^h]$ in more than one way

$$[M_1,\ M_2] \stackrel{SU(2)}{\longrightarrow} \cdots \oplus [J^h] \oplus \cdots \oplus [J^h] \oplus \cdots\,. \tag{3.4.47}$$

The cases of interest for us are $J^h = 0,\ 1/2,\ 1$. $J^h = 0$ is contained only in one way

$$J^h = 0\ : \qquad \begin{array}{|c|c|c|c|c|c|} \hline 1 & \cdots & 1 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 \\ \cline{1-3} \end{array} \quad , \tag{3.4.48}$$

while for $J^h = 1/2,\ 1$ we have:

$$J^h = \frac{1}{2}\ : \left\{ \begin{array}{lcl} \zeta = (a) & \rightarrow & \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \cdots & 1 & 3 & \cdots & 3 & i \\ \hline 2 & \cdots & 2 \\ \cline{1-3} \end{array} \\ & & \\ \zeta = (b) & \rightarrow & \begin{array}{|c|c|c|c|c|c|c|} \hline i & 1 & \cdots & 1 & 3 & \cdots & 3 \\ \hline 3 & 2 & \cdots & 2 \\ \cline{1-4} \end{array} \end{array} \right. \tag{3.4.49}$$

$$
J^h = 1 : \begin{cases}
\zeta = (c) \;\rightarrow\; \frac{1}{2}\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & i & j & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & 3 \\ \cline{1-4} \end{array} + (i \leftrightarrow j) \right) \\[2ex]
\zeta = (d) \;\rightarrow\; \begin{array}{|c|c|c|c|c|c|c|c|}\hline i & j & 1 & \cdots & 1 & 3 & \cdots & 3 \\ \hline 3 & 3 & 2 & \cdots & 2 \\ \cline{1-5} \end{array} \\[2ex]
\zeta = (e) \;\rightarrow\; \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & 3 & \cdots & 3 & i & j \\ \hline 2 & \cdots & 2 \\ \cline{1-3} \end{array}
\end{cases} \tag{3.4.50}
$$

3.4.4 The constraints on the irreducible representations

As I said, the expansion of a generic field contains only the harmonics whose H- and G-quantum numbers are such that the G representation, decomposed under H, contains the H representation of the field. This fact poses some constraints on the G-quantum numbers.

Depending on which constraints are satisfied by a certain G representation, only part of the harmonics is present, and only their corresponding four–dimensional fields appear in the spectrum. Then, in the G representations in which such field disappear, there is *multiplet shortening*. In the modern perspective of Kaluza Klein theory, the exact spectrum of the short multiplets is crucial. Hence the importance of analyzing this disappearance of harmonics with care.

Every harmonic is defined by its $SU(2) \times U(1)' \times U(1)''$ representation, identified by the labels $[J^h\ Z'\ Z'']$. Substituting these values in equations (3.2.12), (3.2.13),

$$
\begin{aligned}
\mathrm{i}\sqrt{3}\lambda_8 + 2\mathrm{i}\left(\frac{\mathrm{i}}{2}\sigma_3\right) - 4\mathrm{i}Y &= Z' \\
-\frac{\mathrm{i}}{2}\sqrt{3}\lambda_8 + 3\mathrm{i}\left(\frac{\mathrm{i}}{2}\sigma_3\right) - 4\mathrm{i}Y &= Z'' \,,
\end{aligned} \tag{3.4.51}
$$

we can determine the constraints of the G representations.

The eigenvalue of $\sqrt{3}\lambda_8 = \mathrm{diag}(1,1,-2)$ depends on ζ: it is $2(M_2 - M_1)$ for the scalar, and

$$
\begin{aligned}
\zeta = (a) :& \qquad \sqrt{3}\lambda_8 = 2(M_2 - M_1) + 3 \\
\zeta = (b) :& \qquad \sqrt{3}\lambda_8 = 2(M_2 - M_1) - 3 \\
\zeta = (c) :& \qquad \sqrt{3}\lambda_8 = 2(M_2 - M_1) \\
\zeta = (d) :& \qquad \sqrt{3}\lambda_8 = 2(M_2 - M_1) - 6 \\
\zeta = (e) :& \qquad \sqrt{3}\lambda_8 = 2(M_2 - M_1) + 6 \,.
\end{aligned} \tag{3.4.52}
$$

Simplifying $\frac{1}{2}\sigma_3$ one finds the first constraint, that is the value of Y in terms of M_1 and M_2. In the cases of interest for us we have five possible expressions of Y, identifying five families of G representations which we denote with the superscripts ^{0}, ${}^+$, ${}^-$, ${}^{++}$ and ${}^{--}$:

$$
\begin{aligned}
&\left.\begin{aligned}
{}^0 : \quad Y &= 2/3(M_2 - M_1) \\
{}^{++} : \quad Y &= 2/3(M_2 - M_1) - 2 \\
{}^{--} : \quad Y &= 2/3(M_2 - M_1) + 2
\end{aligned}\right\} \quad \text{for bosonic fields} \\
&\left.\begin{aligned}
{}^+ : \quad Y &= 2/3(M_2 - M_1) - 1 \\
{}^- : \quad Y &= 2/3(M_2 - M_1) + 1
\end{aligned}\right\} \quad \text{for fermionic fields .}
\end{aligned} \tag{3.4.53}
$$

It is worth noting that the value of Y identifies a $U(1)_R$ representation, so these five families of representations correspond to the five possible representations of $U(1)_R$.

The second constraint arising from (3.4.51) is the lower bound on the quantum number J, since its third component $J_3 = \frac{1}{2}\sigma_3$ is linked to Y. We have three possibilities:

$$J \geq \begin{cases} |Y/2| \\ |Y/2+1| \\ |Y/2-1| \, . \end{cases} \tag{3.4.54}$$

The last kind of constraint refers to M_1 and M_2. If M_1, M_2 are too small, the decomposition (3.4.47) can contain less $[J^h]$ representations, because not all of the Young tableaux (3.4.48), (3.4.49), (3.4.50) do exist. The conditions for the existence of the representations $[J^h]_\zeta$ in $[M_1,\, M_2]$ are:

constraints	J^h	ζ
$M_1 \geq 0$ $M_2 \geq 0$	0	$-$
$M_1 \geq 1$ $M_2 \geq 0$	$\frac{1}{2}$	(a)
$M_1 \geq 0$ $M_2 \geq 1$	$\frac{1}{2}$	(b)
$M_1 \geq 1$ $M_2 \geq 1$	1	(c)
$M_1 \geq 0$ $M_2 \geq 2$	1	(d)
$M_1 \geq 2$ $M_2 \geq 0$	1	(e)

(3.4.55)

We organize the series of the $G = G' \times U(1)_R$ representations in the following way. The constraints (3.4.54) and (3.4.55), with the five values of Y in terms of $M_1,\ M_2$ given by (3.4.53), define the series of G' representations that we list in table 3.1. Every G' representation, together with a superscript ^{0}, $^+$, $^-$, $^{++}$ or $^{--}$ that define the value of Y, is a G representation. So the series of G' representations defined in table 3.1 with such a superscript are series of representations of the whole G group.

For each family of representations (^{0}, $^+$, $^-$, $^{++}$, and $^{--}$) we call a series *regular* if it contains the maximum number of harmonics. The regular series cover all the representations with M_1, M_2 and J sufficiently high to satisfy all the inequality constraints. When some of these inequalities are not satisfied instead, some of the harmonics may be absent in the expansion. The series $A_R, A_1, \ldots, A_8$ are defined by means of the constraints arising in the cases ^{0}, $^+$, $^-$, while the series $B_R, B_1, \ldots, B_{11}$ are defined by means of the constraints arising in the cases $^{++}$, $^{--}$.

In tables 3.2, 3.3, 3.4, 3.5, 3.6, we show which harmonics are present for the different series of G representations. The first column contains the name of each series. The other columns contain the possible harmonics, each labeled by its H-quantum numbers. An asterisk denotes the presence of a given harmonic. To obtain the constraints on the conjugate series it suffices to exchange M_1 and M_2, as explained in [21], [37].

Tables 3.1,…3.6 are the results of the analysis of equations (3.4.51) for the relevant $[J^h, Z', Z'']$ values.

G'−name	$M_1,\ M_2$	J constraints
A_R	$M_2 > 0,\ M_1 > 0$	$J > \|(M_2 - M_1)/3\|$
A_1	$M_2 > M_1 > 0$	$J = (M_2 - M_1)/3$
A_2	$M_2 > M_1 > 0$	$J = (M_2 - M_1)/3 - 1$
A_3	$M_2 > M_1 = 0$	$J > (M_2 - M_1)/3$
A_4	$M_2 > M_1 = 0$	$J = (M_2 - M_1)/3$
A_5	$M_2 > M_1 = 0$	$J = (M_2 - M_1)/3 - 1$
A_6	$M_2 = M_1 > 0$	$J = 0$
A_7	$M_2 = M_1 = 0$	$J > 0$
A_8	$M_2 = M_1 = 0$	$J = 0$
B_R	$M_2 > 1,\ M_1 \geq 0$	$J > \|(M_2 - M_1)/3 - 1\|$
B_1	$M_2 > M_1 + 3$	$J = (M_2 - M_1)/3 - 1$
B_2	$M_2 > M_1 + 3$	$J = (M_2 - M_1)/3 - 2$
B_3	$M_2 = M_1 + 3$	$J = (M_2 - M_1)/3 - 1$
B_4	$M_1 \geq M_2 > 1$	$J = -(M_2 - M_1)/3 + 1$
B_5	$M_1 \geq M_2 > 1$	$J = -(M_2 - M_1)/3$
B_6	$M_1 \geq M_2 = 1$	$J > -(M_2 - M_1)/3 + 1$
B_7	$M_1 \geq M_2 = 1$	$J = -(M_2 - M_1)/3 + 1$
B_8	$M_1 \geq M_2 = 1$	$J = -(M_2 - M_1)/3$
B_9	$M_1 \geq M_2 = 0$	$J > -(M_2 - M_1)/3 + 1$
B_{10}	$M_1 \geq M_2 = 0$	$J = -(M_2 - M_1)/3 + 1$
B_{11}	$M_1 \geq M_2 = 0$	$J = -(M_2 - M_1)/3$

Table 3.1: Series of G' representations

J^h	0	0	0	1/2	1/2	1/2	1/2	1/2	1/2	1	1	1
Z'	0	$-2i$	$2i$	$3i$	$-3i$	i	$-i$	$5i$	$-5i$	0	$-2i$	$2i$
Z''	0	$-3i$	$3i$	$-3/2i$	$3/2i$	$-9/2i$	$9/2i$	$3/2i$	$-3/2i$	0	$-3i$	$3i$
μ				(a)	(b)	(a)	(b)	(a)	(b)	(c)	(c)	(c)
$A_R^{\,0}$	*	*	*	*	*	*	*	*	*	*	*	*
$A_1^{\,0}$	*	*		*	*	*			*	*	*	
$A_1^{*\,0}$	*		*	*	*		*	*		*		*
$A_2^{\,0}$		*				*			*		*	
$A_2^{*\,0}$			*				*	*				*
$A_3^{\,0}$	*	*	*		*		*		*			
$A_3^{*\,0}$	*	*	*	*		*		*				
$A_4^{\,0}$	*	*			*				*			
$A_4^{*\,0}$	*		*	*				*				
$A_5^{\,0}$		*							*			
$A_5^{*\,0}$			*					*				
$A_6^{\,0}$	*			*	*					*		
$A_7^{\,0}$	*	*	*									
$A_8^{\,0}$	*											

Table 3.2: Harmonics content for the series of type ^{0}

J^h	0	0	1/2	1/2
Z'	$2i$	$4i$	$-i$	i
Z''	$-3i$	0	$-3i/2$	$3i/2$
μ			(b)	(b)
A_R^+	$*$	$*$	$*$	$*$
A_1^+	$*$	$*$	$*$	$*$
A_1^{*+}		$*$		$*$
A_2^+	$*$		$*$	
A_2^{*+}				
A_3^+	$*$	$*$	$*$	$*$
A_3^{*+}	$*$	$*$		
A_4^+	$*$	$*$	$*$	$*$
A_4^{*+}		$*$		
A_5^+	$*$		$*$	
A_5^{*+}				
A_6^+		$*$		$*$
A_7^+	$*$	$*$		
A_8^+		$*$		

Table 3.3: Harmonics content for the series of type $^+$

J^h	0	0	1/2	1/2
Z'	$-2i$	$-4i$	i	$-i$
Z''	$3i$	0	$3i/2$	$-3i/2$
μ			(a)	(a)
A_R^-	$*$	$*$	$*$	$*$
A_1^-		$*$		$*$
A_1^{*-}	$*$	$*$	$*$	$*$
A_2^-				
A_2^{*-}	$*$		$*$	
A_3^-	$*$	$*$		
A_3^{*-}	$*$	$*$	$*$	$*$
A_4^-		$*$		
A_4^{*-}	$*$	$*$	$*$	$*$
A_5^-				
A_5^{*-}	$*$		$*$	
A_6^-		$*$		$*$
A_7^-	$*$	$*$		
A_8^-		$*$		

Table 3.4: Harmonics content for the series of type $^-$

J^h	0	0	0	1/2	1/2	1/2	1	1	1
Z'	$6i$	$8i$	$4i$	$3i$	i	$5i$	0	$-2i$	$2i$
Z''	$-3i$	0	$-6i$	$-3/2i$	$-9/2i$	$3/2i$	0	$-3i$	$3i$
μ				(b)	(b)	(b)	(d)	(d)	(d)
${B_R}^{++}$	*	*	*	*	*	*	*	*	*
${B_1}^{++}$	*		*	*	*		*	*	
${B_2}^{++}$			*		*			*	
${B_3}^{++}$	*			*			*		
${B_4}^{++}$	*	*		*		*	*		*
${B_5}^{++}$		*				*			*
${B_6}^{++}$	*	*	*	*	*	*			
${B_7}^{++}$	*	*		*		*			
${B_8}^{++}$		*				*			
${B_9}^{++}$	*	*	*						
${B_{10}}^{++}$	*	*							
${B_{11}}^{++}$		*							

Table 3.5: Harmonics content for the series of type $^{++}$

J^h	0	0	0	1/2	1/2	1/2	1	1	1
Z'	$-6i$	$-8i$	$-4i$	$-3i$	$-i$	$-5i$	0	$2i$	$-2i$
Z''	$3i$	0	$6i$	$3/2i$	$9/2i$	$-3/2i$	0	$3i$	$-3i$
μ				(a)	(a)	(a)	(e)	(e)	(e)
$B_R^{*\,--}$	*	*	*	*	*	*	*	*	*
$B_1^{*\,--}$	*		*	*	*		*	*	
$B_2^{*\,--}$			*		*			*	
$B_3^{*\,--}$	*			*			*		
$B_4^{*\,--}$	*	*		*		*	*		*
$B_5^{*\,--}$		*				*			*
$B_6^{*\,--}$	*	*	*	*	*	*			
$B_7^{*\,--}$	*	*		*		*			
$B_8^{*\,--}$		*				*			
$B_9^{*\,--}$	*	*	*						
$B_{10}^{*\,--}$	*	*							
$B_{11}^{*\,--}$		*							

Table 3.6: Harmonics content for the series of type $^{--}$

3.4.5 Differential calculus via harmonic analysis

The Kaluza Klein kinetic operators $\boxtimes_y^{[\lambda_1\lambda_2\lambda_3]}$ act as finite dimensional matrices on the harmonic subspaces of fixed G-quantum numbers:

$$\boxtimes_y^{[\lambda_1\lambda_2\lambda_3]} e_\xi^{[M_1M_2JY]}(y) = \mathcal{M}\left([M_1M_2JY]\right)_\xi{}^{\xi'}\, e_{\xi'}^{[M_1M_2JY]}(y) \tag{3.4.56}$$

(see (3.3.28), (3.3.29)).

Let us now consider the explicit action of the covariant derivative (3.1.26) on the harmonics. It is given by the (3.1.27)

$$\mathcal{D} = d + \Omega^H t_H + \mathcal{B}^a \mathbb{M}_a \equiv \mathcal{D}^H + \mathcal{B}^a \mathbb{M}_a\,, \tag{3.4.57}$$

where t_H are the generators of H and $\mathbb{M}_a$ the part of the $SO(7)$-connection not belonging to H. The zero-forms transform in the trivial representation of the tangent space structure group $SO(7)$. In other words, the $SO(7)$ generators in the scalar representation vanish identically. This means that the covariant derivatives equal the simple one: $\mathcal{D} = \mathcal{D}^H = d$. For the vector representation instead, we can easily compute the matrices $(\mathbb{M}_c)^a{}_b$ from eq. (3.1.26):

$$\mathbb{M}_1 = \begin{array}{c}\left(\begin{array}{cc|c|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ \hline 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \underbrace{\hspace{2.5em}}_{m}\underbrace{\hspace{1.5em}}_{3}\underbrace{\hspace{5em}}_{A}\end{array}, \qquad \mathbb{M}_2 = \left(\begin{array}{cc|c|cccc} 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathbb{M}_3 = \frac{1}{3}\left(\begin{array}{cc|c|cccc} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathbb{M}_4 = \left(\begin{array}{cc|c|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \mathbb{M}_5 = \left(\begin{array}{cc|c|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$
\mathbb{M}_6 = \left(\begin{array}{cc|c|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{array} \right), \qquad \mathbb{M}_7 = \left(\begin{array}{cc|c|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right). \tag{3.4.58}
$$

We know that (3.3.16)

$$
\mathcal{D}^H_a \left(\mathcal{H}^i{}_n\right) = -r_a \left(T_a \mathcal{H}\right)^i{}_n = -r_a \left(T_a\right)^i_m \mathcal{H}^m{}_n. \tag{3.4.59}
$$

By means of eq. (3.2.39) we can calculate the explicit components of $\mathcal{D}^H$, i.e. its projection along the vielbein:

$$
\begin{gathered}
\mathcal{D}^H = \mathcal{B}^a \mathcal{D}^H_a = -\Omega^a t_a, \\
\left\{ \begin{array}{lcl} \mathcal{D}^H_A & = & -\frac{4}{\sqrt{3}} i \lambda_A, \\ \mathcal{D}^H_m & = & -\frac{4}{\sqrt{2}} i \sigma_m, \\ \mathcal{D}^H_3 & = & -\frac{4}{3} Z, \end{array} \right.
\end{gathered} \tag{3.4.60}
$$

where the coset generators t_a act on the harmonics as follows. λ_A acts on the $SU(3)$ part of the G representation of the harmonic. The fundamental representation of λ_A is given by the Gell–Mann matrices (see Appendix A). On a generic Young tableau λ_A acts as the tensor representation.

To give an example, let us consider the case $[M_1, M_2] = [2, 2]$, $[J^h] = [1/2]$, $\zeta = (b)$. The index i in the (3.4.59) (or, more precisely, its $SU(2)$ part) runs in the representation

$$
\begin{array}{|c|c|c|c|} \hline 1 & i & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \;. \tag{3.4.61}
$$

With this notation we write the H representation as a fragment of the G representation (in the (3.4.59) the index of the G representation is m). Let us consider the component $i = 1$ of this representation,

$$
\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \;. \tag{3.4.62}
$$

We can determine from (3.4.59) the action of

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{3.4.63}
$$

on this component:

$$
\begin{aligned}
\lambda_4 & \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} = \\
&= \begin{array}{|c|c|c|c|} \hline 3 & 1 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 1 \\ \cline{1-2} \end{array} + 2 \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} = \\
&\qquad = \begin{array}{|c|c|c|c|} \hline 3 & 1 & 3 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} + 2 \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} \end{array} \;.
\end{aligned} \tag{3.4.64}
$$

Similarly, σ^m ($m = \{1,2\}$) acts as the m-th Pauli matrix on the fundamental representation of $SU(2)$, and as its n-th tensor power on the n-boxes $SU(2)$ Young tableau:

$$\sigma^1 \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline\end{array} = 3 \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&2\\\hline\end{array} + 2 \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&2\\\hline\end{array} . \tag{3.4.65}$$

Finally, Z acts trivially, multiplying the harmonic by its Z-charge:

$$Z = \frac{\mathrm{i}}{2}\sqrt{3}\lambda_8 + \frac{\mathrm{i}}{2}\sigma_3 + \mathrm{i}Y \tag{3.4.66}$$

$$\begin{aligned} Z \ &\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&3\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline\end{array} = \\ &= \left(-\frac{3}{2}i - \frac{1}{2}i + iY\right) \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&3\\\cline{1-2}\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline\end{array} . \end{aligned} \tag{3.4.67}$$

In the course of the calculations, one often encounters the H-covariant Laplace-Beltrami operator on G/H:

$$\delta^{ab}\mathcal{D}^H_a\mathcal{D}^H_b = \frac{16}{3}\lambda_A\lambda_A + \frac{16}{2}\sigma^m\sigma^m - \frac{16}{9}Z^2 . \tag{3.4.68}$$

The eigenvalues of the first operator, $\lambda_A\lambda_A$, are listed in the following table:

J^h	ζ	$\lambda^A\lambda^A$ eigenvalues
0	$-$	$4(M_1 + M_2 + M_1M_2)$
1/2	(a)	$2(4M_1 + 2M_1M_2 - 3)$
1/2	(b)	$2(4M_2 + 2M_1M_2 - 3)$
1	(c)	$4(M_1 + M_2 + M_1M_2 - 2)$
1	(d)	$4(3M_2 - M_1 + M_1M_2 - 5)$
1	(e)	$4(3M_1 - M_2 + M_1M_2 - 5)$

while the eigenvalues of $\sigma^m\sigma^m$ depend on the $SU(2)$ quantum numbers J and J_3:

$$\sigma^m\sigma^m \ \underbrace{\begin{array}{|c|c|c|}\hline 1&\cdots&1\\\hline\end{array}}_{m_1}\underbrace{\begin{array}{|c|c|c|}\hline 2&\cdots&2\\\hline\end{array}}_{m_2} = 4\left[J(J+1) - J_3^2\right] \begin{array}{|c|c|c|c|c|c|}\hline 1&\cdots&1&2&\cdots&2\\\hline\end{array} , \tag{3.4.69}$$

where $2J = m_1 + m_2$ and $2J_3 = m_1 - m_2$. The complete Kaluza Klein mass operator heavily depends on the kind of field it acts on and will be analyzed in detail in the next sections.

The zero-form

The only representation into which the $[0,0,0]$ (i.e. the scalar) of $SO(7)$ breaks under H, is obviously the H-scalar representation. The question now is: which G-irreducible representations do contain the H-scalar? From equations (3.4.51) we see that $Z' = Z'' = 0$ implies

$$2J_3 = Y = \frac{2}{3}(M_2 - M_1) . \tag{3.4.70}$$

This means that

- $M_2 - M_1 \in 3\mathbb{Z}$
- $J \in \mathbb{N}$
- $J \geq \left|\frac{1}{3}(M_2 - M_1)\right|$
- $Y = \frac{2}{3}(M_2 - M_1)$.

We will denote the scalar as

$$\mathcal{Y}(x,y) = [0|\mathrm{I}](x,y) \equiv \sum\nolimits'_{[M_1 M_2 J Y]} \mathcal{H}^{[M_1 M_2 JY]}_{[000]}(y) \cdot S_{[M_1 M_2 Y J]}(x). \tag{3.4.71}$$

The Kaluza Klein mass operator for the zero-form $\mathcal{Y}$ is given by

$$\boxtimes^{[000]}\mathcal{Y} \equiv \mathcal{D}_b \mathcal{D}^b \mathcal{Y} = \mathcal{D}^H_b \mathcal{D}^{Hb}\mathcal{Y}. \tag{3.4.72}$$

For the scalar, there are no $\mathbb{M}$–connection terms. So, by means of eq. (3.4.68), the computation of its eigenvalues, on the G–representations as listed above, is immediate:

$$\begin{aligned}\boxtimes^{[000]}\mathcal{Y} \equiv M_{(0)^3}\mathcal{Y} &= \left[\tfrac{64}{3}(M_1 + M_2 + M_1 M_2) + 32J(J+1) + \tfrac{32}{9}(M_2 - M_1)^2\right]\mathcal{Y} = \\ &= H_0 \mathcal{Y}\end{aligned} \tag{3.4.73}$$

where H_0 is the same quantity defined in eq. (3.4.5).

As we see from the Kaluza Klein expansion (3.1.61), the eigenvalues of the zero–form harmonic allow us to determine the masses of the AdS_4 graviton field h and the scalar fields S, Σ.

The one-form

Let us decompose under H the vector representation of $SO(7)$. The generators of H in this representation are given by (3.4.42), (3.4.43), (3.4.44). We see that

$$\mathbf{7} \longrightarrow \mathbf{4} \oplus \mathbf{2} \oplus \mathbf{1}\,. \tag{3.4.74}$$

It is convenient to move to a complex basis. The real four dimensional representation of $SU(2)$ is a complex two dimensional representation, and the real two dimensional representation is a complex one dimensional representation. The change from the real to the complex basis can be performed as follows (see also [8]):

$$\begin{aligned}\mathcal{Y}^A &= \lambda^A_{3i}\langle 1|\mathrm{I}\rangle_i + \lambda^A_{i3}\langle 1|\mathrm{I}\rangle^*_i && (3.4.75)\\ \mathcal{Y}^m &= \sigma^m_{21}\langle 1|\mathrm{I}\rangle_. + \sigma^m_{12}\langle 1|\mathrm{I}\rangle^*_. && (3.4.76)\\ \mathcal{Y}^3 &= [1,\mathrm{I}]\,. && (3.4.77)\end{aligned}$$

By applying (3.4.42), (3.4.43), (3.4.44) on these fragments one finds the Z', Z'' eigenvalues.

As a result of this calculation, one finds that the decomposition under H of the vector representation of $SO(7)$ is the following [10]:

$$[1,0,0] \to [0,0,0] \oplus [0,-2i,-3i] \oplus [0,2i,3i] \oplus [\tfrac{1}{2},3i,-\tfrac{3}{2}i] \oplus [\tfrac{1}{2},-3i,\tfrac{3}{2}i]. \tag{3.4.78}$$

[10]When we write a pair of complex conjugate representations we assume a conjugation relation between them. For example, by writing $[0,-2i,-3i] \oplus [0,2i,3i]$ we intend a complex representation of complex dimension one or real dimension two.

These H–irreducible fragments can be expanded as in (3.4.46) [11] (summation over the G-quantum numbers is intended):

For type ^{0} :

$$
\begin{aligned}
\langle 1|\mathrm{I}\rangle_i &= \mathcal{H}_i^{[1/2,3i,-3i/2](a)} \cdot W\langle \tfrac{1}{2},\mathrm{I}\rangle\,, \\
\langle 1|\mathrm{I}\rangle_i^* &= \varepsilon^{ij}\mathcal{H}_j^{[1/2,-3i,3i/2](b)} \cdot \widetilde{W}\langle \tfrac{1}{2},\mathrm{I}\rangle\,, \\
\langle 1|\mathrm{I}\rangle_\cdot &= \mathcal{H}^{[0,-2i,-3i]} \cdot W\langle 0,\mathrm{I}\rangle\,, \\
\langle 1|\mathrm{I}\rangle_\cdot^* &= \mathcal{H}^{[0,2i,3i]} \cdot \widetilde{W}\langle 0,\mathrm{I}\rangle\,, \\
[1|\mathrm{I}]_\cdot &= \mathcal{H}^{[0,0,0]} \cdot W[0,\mathrm{I}]\,,
\end{aligned}
\tag{3.4.79}
$$

For type $^{++}$:

$$
\langle 1|\mathrm{I}\rangle_i = \mathcal{H}_i^{[1/2,3i,-3/2i](b)} \cdot W\langle \tfrac{1}{2},\mathrm{II}\rangle\,,
$$

For type $^{--}$:

$$
\langle 1|\mathrm{I}\rangle_i^* = -\varepsilon^{ij}\mathcal{H}_j^{[1/2,-3i,3/2i](a)} \cdot \widetilde{W}\langle \tfrac{1}{2},\mathrm{II}\rangle\,.
$$

As we see, there are five different AdS_4 fields $(W,\widetilde{W})$ in the case of the ^{0} series, and one field in the case of the $^{++}$ and $^{--}$ series. So, for the regular ^{0} series the Laplace Beltrami operator acts on the AdS_4 fields as a 5×5 matrix. For the *exceptional* series it acts as a matrix of lower dimension.

The Laplace Beltrami operator for the transverse one-form field $\mathcal{Y}^a$, is given by

$$
\boxtimes^{[100]}\mathcal{Y}^a \equiv M_{(1)(0)^2}\mathcal{Y}^a = 2\mathcal{D}_b\mathcal{D}^{[b}\mathcal{Y}^{a]} = (\mathcal{D}^b\mathcal{D}_b + 24)\mathcal{Y}^a\ , \tag{3.4.80}
$$

where transversality of $\mathcal{Y}^a$ means that $\mathcal{D}_a\mathcal{Y}^a = 0$. From the decomposition $\mathcal{D}_a = \mathcal{D}^H_a + \mathbb{M}_a$ we obtain:

$$
\boxtimes^{[100]}\mathcal{Y}^a = (\mathcal{D}^{Hb}\mathcal{D}^H_b + 24)\mathcal{Y}^a + \eta^{gd}\left(2(\mathbb{M}_g)^a{}_b\mathcal{D}^H_d + (\mathbb{M}_g)^a{}_e(\mathbb{M}_d)^e{}_b\right)\mathcal{Y}^b\,. \tag{3.4.81}
$$

The matrix of this operator on the AdS_4 fields is given by

$M_{(1)(0)^2}$	$W\langle\frac{1}{2},\mathrm{I}\rangle$	$\widetilde{W}\langle\frac{1}{2},\mathrm{I}\rangle$	$W\langle 0,\mathrm{I}\rangle$	$\widetilde{W}\langle 0,\mathrm{I}\rangle$	$W[0,\mathrm{I}]$
$W\langle\frac{1}{2},\mathrm{I}\rangle$	$H_0-\frac{32(M_2-M_1)}{3}$	0	0	0	$\frac{16M_1}{\sqrt{3}}$
$\widetilde{W}\langle\frac{1}{2},\mathrm{I}\rangle$	0	$H_0+\frac{32(M_2-M_1)}{3}$	0	0	$\frac{16M_2}{\sqrt{3}}$
$W\langle 0,\mathrm{I}\rangle$	0	0	$H_0+\frac{32(M_2-M_1)}{3}$	0	$-\frac{8(2J+Y)}{\sqrt{2}}$
$\widetilde{W}\langle 0,\mathrm{I}\rangle$	0	0	0	$H_0-\frac{32(M_2-M_1)}{3}$	$\frac{8(2J-Y)}{\sqrt{2}}$
$W[0,\mathrm{I}]$	$\frac{32(2+M_2)}{\sqrt{3}}$	$\frac{32(2+M_1)}{\sqrt{3}}$	$-\frac{16(2+2J-Y)}{\sqrt{2}}$	$\frac{16(2+2J+Y)}{\sqrt{2}}$	H_0+48

(3.4.82)

[11] Using the same conventions as in [21], [22], [37], [8], the reader might notice that there appears a sign $(-1)^{J-J_3}$ upon taking the complex conjugate of the fragments $\langle\ldots|\ldots\rangle_x$. In order to reduce the notation we have absorbed this sign in the x–space fields $\widetilde{W}\langle\ldots,\ldots\rangle$. This will be done for all the complex conjugates henceforth.

Its eigenvalues are:

$$\begin{aligned}
\lambda_1 &= H_0 + \tfrac{32}{3}(M_2 - M_1)\,, \\
\lambda_2 &= H_0 - \tfrac{32}{3}(M_2 - M_1)\,, \\
\lambda_3 &= H_0\,, \\
\lambda_4 &= H_0 + 24 + 4\sqrt{H_0 + 36}\,, \\
\lambda_5 &= H_0 + 24 - 4\sqrt{H_0 + 36}\,.
\end{aligned} \tag{3.4.83}$$

Actually, what we have just calculated are the eigenvalues of

$$M_{(1)(0)^2}\mathcal{Y}^a + \mathcal{D}^a\mathcal{D}_b\mathcal{Y}^b\,. \tag{3.4.84}$$

It coincides with $M_{(1)(0)^2}$ when acting on a transverse one-form. But on a generic $\mathcal{Y}^a$, which possibly contains a longitudinal term, the second part of (3.4.84), $\mathcal{D}^a\mathcal{D}_b\mathcal{Y}^b$, is not inert. Indeed, let us suppose

$$\mathcal{Y}^a = \mathcal{D}^a\mathcal{Y}$$

for some scalar function $\mathcal{Y}$. Then

$$\mathcal{D}^a\mathcal{D}_b\mathcal{Y}^b = \mathcal{D}^a\mathcal{D}_b\mathcal{D}^b\mathcal{Y} = \mathcal{D}^a M_{(0)^3}\mathcal{Y} = M_{(0)^3}\mathcal{Y}^a. \tag{3.4.85}$$

So, our actual operator (3.4.82) contains the eigenvalues of $M_{(0)^3}$, which are *longitudinal* (hence *non-physical*) for the one–form. This fact is true also for the two-form.

The eigenvalue λ_3 in (3.4.83) is the longitudinal one, equal to the zero–form eigenvalue H_0. The other four, instead, are transverse physical eigenvalues.

The matrices corresponding to the exceptional series are easily obtained from (3.4.82) by removing the rows and the columns of the fields that disappear in the expansions (3.4.79), as we read from table 3.2. We list the mass eigenvalues of each series:

A_R	$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$
A_1	$\lambda_1, \lambda_3, \lambda_4, \lambda_5$
A_1^*	$\lambda_2, \lambda_3, \lambda_4, \lambda_5$
A_2	λ_1
A_2^*	λ_2
A_3	$\lambda_1, \lambda_3, \lambda_4, \lambda_5$
A_3^*	$\lambda_2, \lambda_3, \lambda_4, \lambda_5$
A_4	$\lambda_1, \lambda_3, \lambda_4$
A_4^*	$\lambda_2, \lambda_3, \lambda_4$
A_5	λ_1
A_5^*	λ_2
A_6	$\lambda_3, \lambda_4, \lambda_5$
A_7	$\lambda_3, \lambda_4, \lambda_5$
A_8	λ_4

(3.4.86)

For the series of type $^{++}$ the operator $M_{(1)(0)^2}$ acts as a 1×1 matrix on the AdS_4 fields and has eigenvalue:

$$H_0 + \tfrac{32}{3}(M_2 - M_1) \tag{3.4.87}$$

for the series B_R, B_1, B_3, B_4, B_6 and B_7. For the type $^{--}$-series the eigenvalue is the conjugate one ($M_2 \leftrightarrow M_1$) in the conjugate series.

We can use the eigenvalues of the one–form harmonic to determine (see next section and [22]) the masses of the AdS_4vector field A, W.

The two-form

Under the action of $H = SU(2)\times U(1)'\times U(1)''$ the 21 components of the $SO(7)$ two-form transform into the completely reducible representation:

$$\begin{aligned}[1,1,0] \quad \rightarrow \quad & [1,0,0]\oplus[0,0,0]\oplus[0,0,0]\oplus[0,6i,-3i]\oplus[0,-6i,3i]\oplus \\ & \oplus[1/2,i,-9/2i]\oplus[1/2,-i,9/2i]\oplus[1/2,5i,3/2i]\oplus[1/2,-5i,-3/2i] \\ & \oplus[1/2,3i,-3/2i]\oplus[1/2,-3i,3/2i]\oplus[0,-2i,-3i]\oplus[0,2i,3i]\,. \end{aligned} \tag{3.4.88}$$

The decomposition of the two–form in H–irreducible fragments is as follows:

$$\begin{aligned}
\mathcal{Y}^{AB} &= -i\lambda^{[A}_{\ i3}\lambda^{B]}_{\ 3i}\,[2|\mathrm{I}]_{.} - i\lambda^{[A}_{\ i3}\,\lambda^{B]}_{\ 3j}\,\varepsilon^{ik}[2|\mathrm{I}]_{jk} + \\
&\quad \lambda^{[A}_{\ i3}\lambda^{B]}_{\ 3j}\,\varepsilon^{ik}\langle 2|\mathrm{I}\rangle_{jk} + \lambda^{[A}_{\ 3i}\lambda^{B]}_{\ j3}\,\varepsilon^{ik}\langle 2|\mathrm{I}\rangle^{*}_{jk} + \\
&\quad \lambda^{[A}_{\ 3i}\lambda^{B]}_{\ 3j}\,\varepsilon^{ij}\langle 2|\mathrm{I}\rangle_{.} + \lambda^{[A}_{\ i3}\lambda^{B]}_{\ j3}\,\varepsilon^{ij}\langle 2|\mathrm{I}\rangle^{*}_{.} \\
\mathcal{Y}^{Am} &= \lambda^{A}_{\ 3i}\,\sigma^{m}_{\ 21}\langle 2|\mathrm{II}\rangle_{i} + \lambda^{A}_{\ i3}\,\sigma^{m}_{\ 12}\langle 2|\mathrm{II}\rangle^{*}_{i} + \\
&\quad \lambda^{A}_{\ 3i}\sigma^{m}_{\ 12}\langle 2|\mathrm{III}\rangle_{i} + \lambda^{A}_{\ i3}\sigma^{m}_{\ 21}\langle 2|\mathrm{III}\rangle^{*}_{i} \\
\mathcal{Y}^{mn} &= \varepsilon^{mn}[2|\mathrm{II}]_{.} \\
\mathcal{Y}^{m3} &= \sigma^{m}_{\ 21}\langle 2|\mathrm{II}\rangle_{.} + \sigma^{m}_{\ 12}\langle 2|\mathrm{II}\rangle^{*}_{.} \\
\mathcal{Y}^{A3} &= \lambda^{A}_{\ 3i}\langle 2|\mathrm{I}\rangle_{i} + \lambda^{A}_{\ i3}\langle 2|\mathrm{I}\rangle^{*}_{i}\,,
\end{aligned}$$

where:

$$\begin{aligned}
[2|\mathrm{I}]_{.} &= \mathcal{H}^{[0,0,0]}\;Z[0,\mathrm{I}|\rho] \\
[2|\mathrm{II}]_{.} &= \mathcal{H}^{[0,0,0]}\;Z[0,\mathrm{II}|\rho] \\
\langle 2|\mathrm{I}\rangle_{.} &= \mathcal{H}^{[0,6i,-3i]}\;Z\langle 0,\mathrm{I}|\rho\rangle \\
\langle 2|\mathrm{I}\rangle^{*}_{.} &= \mathcal{H}^{[0,-6i,3i]}\;\widetilde{Z}\langle 0,\mathrm{I}|\rho\rangle \\
\langle 2|\mathrm{II}\rangle_{.} &= \mathcal{H}^{[0,-2i,-3i]}\;Z\langle 0,\mathrm{II}|\rho\rangle \\
\langle 2|\mathrm{II}\rangle^{*}_{.} &= \mathcal{H}^{[0,2i,3i]}\;\widetilde{Z}\langle 0,\mathrm{II}|\rho\rangle \\
\langle 2|\mathrm{I}\rangle_{i} &= \mathcal{H}^{[1/2,3i,-3/2i](a)}_{i}\;Z\langle 1/2,\mathrm{I}|\rho\rangle \\
\langle 2|\mathrm{I}\rangle^{*}_{i} &= \varepsilon^{ij}\mathcal{H}^{[1/2,-3i,3/2i](b)}_{j}\;\widetilde{Z}\langle 1/2,\mathrm{I}|\rho\rangle \\
\langle 2|\mathrm{II}\rangle_{i} &= \mathcal{H}^{[1/2,i,-9/2i](a)}_{i}\;Z\langle 1/2,\mathrm{II}|\rho\rangle \\
\langle 2|\mathrm{II}\rangle^{*}_{i} &= \varepsilon^{ij}\mathcal{H}^{[1/2,-i,9/2i](b)}_{j}\;\widetilde{Z}\langle 1/2,\mathrm{II}|\rho\rangle \\
\langle 2|\mathrm{III}\rangle_{i} &= \mathcal{H}^{[1/2,5i,3/2i](a)}_{i}\;Z\langle 1/2,\mathrm{III}|\rho\rangle \\
\langle 2|\mathrm{III}\rangle^{*}_{i} &= \varepsilon^{ij}\mathcal{H}^{[1/2,-5i,-3/2i](b)}_{j}\;\widetilde{Z}\langle 1/2,\mathrm{III}|\rho\rangle \\
[2|\mathrm{I}]_{ij} &= \mathcal{H}^{[1,0,0](c)}_{ij}\;Z[1,\mathrm{I}|\rho] \\
\langle 2|\mathrm{I}\rangle_{ij} &= \mathcal{H}^{[1,0,0](d)}_{ij}\;Z\langle 1,\mathrm{I}|\rho\rangle \\
\langle 2|\mathrm{I}\rangle^{*}_{ij} &= \varepsilon^{ik}\epsilon_{jl}\mathcal{H}^{[1,0,0](e)}_{kl}\;\widetilde{Z}\langle 1,\mathrm{I}|\rho\rangle\,.
\end{aligned}$$

The Laplace Beltrami operator for the transverse two-form field $\mathcal{Y}^{ab}$, is given by

$$\boxtimes^{[110]}\mathcal{Y}^{[ab]} \equiv M_{(1)^2(0)}\mathcal{Y}^{[ab]} = 3\mathcal{D}_{g}\mathcal{D}^{[g}\mathcal{Y}^{ab]} = (\mathcal{D}^{g}\mathcal{D}_{g}+48)\mathcal{Y}^{[ab]} - 4\mathcal{R}^{[a\ \ b]}_{\ \ [g\ \ d]}\mathcal{Y}^{[gd]} \tag{3.4.89}$$

From the decomposition $\mathcal{D}_a\mathcal{Y}^{bg} = \mathcal{D}^H_a\mathcal{Y}^{bg} + (\mathbb{M}_a)^b{}_d\mathcal{Y}^{dg} + (\mathbb{M}_a)^g{}_d\mathcal{Y}^{bd}$ we obtain:

$$\boxtimes^{[110]}\mathcal{Y}^{[ab]} = \Big\{48\,\delta^{[a\ b]}_{[g\ d]} - 4\mathcal{R}^{[a\ b]}_{[g\ d]} +$$

$$+2\eta^{mn}(\mathbb{M}_m)^{[a}{}_{[g}(\mathbb{M}_n)^{b]}{}_{d]} + 2\eta^{mn}(\mathbb{M}_m\mathbb{M}_n)^{[a}{}_{[g}d^{b]}{}_{d]} + 4\eta^{mn}(\mathbb{M}_m)^{[a}{}_{[g}d^{b]}{}_{d]}\mathcal{D}^H_n\Big\}\mathcal{Y}^{[gd]}. \quad (3.4.90)$$

For the regular G representations of type ^{0} this operators acts on AdS_4 fields as the following 11×11 matrix:
Columns one to three:

$M_{(1)^2(0)}$	$Z[0,\mathrm{I}]$	$Z[0,\mathrm{II}]$	$Z[1,\mathrm{I}]$
$Z[0,\mathrm{I}]$	H_0+32	-16	$\frac{16}{\sqrt{3}}i(M_2+2)$
$Z[0,\mathrm{II}]$	-32	H_0+16	0
$Z[1,\mathrm{I}]$	0	0	H_0
$Z\langle\frac{1}{2},\mathrm{I}\rangle$	$-\frac{16}{\sqrt{3}}iM_1$	0	$\frac{16}{\sqrt{3}}i(M_1+2)$
$\widetilde{Z}\langle\frac{1}{2},\mathrm{I}\rangle$	$\frac{16}{\sqrt{3}}iM_2$	0	$-\frac{16}{\sqrt{3}}i(M_2+2)$
$Z\langle 0,\mathrm{II}\rangle$	0	$\frac{16}{3\sqrt{2}}i(M_2-M_1+3J)$	0
$\widetilde{Z}\langle 0,\mathrm{II}\rangle$	0	$-\frac{16}{3\sqrt{2}}i(M_2-M_1-3J)$	0
$Z\langle 0,\mathrm{I}\rangle$	0	0	0
$\widetilde{Z}\langle 0,\mathrm{II}\rangle$	0	0	0
$Z\langle\frac{1}{2},\mathrm{III}\rangle$	0	0	0
$\widetilde{Z}\langle\frac{1}{2},\mathrm{III}\rangle$	0	0	0

Columns four to seven:

$M_{(1)^2(0)}$	$Z\langle\frac{1}{2},\mathrm{I}\rangle$	$\widetilde{Z}\langle\frac{1}{2},\mathrm{I}\rangle$	$Z\langle 0,\mathrm{II}\rangle$	$\widetilde{Z}\langle 0,\mathrm{II}\rangle$
$Z[0,\mathrm{I}]$	$-\frac{16}{\sqrt{3}}i(M_1+2)$	0	0	0
$Z[0,\mathrm{II}]$	0	0	$\frac{32}{2\sqrt{2}}i(M_2-M_1-3J)$	$-\frac{32}{2\sqrt{2}}i(M_2-M_1-3J)$
$Z[1,\mathrm{I}]$	$-\frac{16}{\sqrt{3}}iM_2$	$\frac{16}{\sqrt{3}}iM_1$	0	0
$Z\langle\frac{1}{2},\mathrm{I}\rangle$	$H_0+32-\frac{32}{3}(M_2-M_1)$	0	0	0
$\widetilde{Z}\langle\frac{1}{2},\mathrm{I}\rangle$	0	$H_0+32+\frac{32}{3}(M_2-M_1)$	0	0
$Z\langle 0,\mathrm{II}\rangle$	0	0	$H_0+32+\frac{32}{3}(M_2-M_1)$	0
$\widetilde{Z}\langle 0,\mathrm{II}\rangle$	0	0	0	$H_0+32-\frac{32}{3}(M_2-M_1)$
$Z\langle 0,\mathrm{I}\rangle$	$-\frac{8}{3\sqrt{2}}(M_2-M_1+3J)$	0	$-\frac{16}{\sqrt{3}}M_1$	0
$\widetilde{Z}\langle 0,\mathrm{II}\rangle$	0	$-\frac{16}{3\sqrt{2}}(M_2-M_1-3J)$	0	$-\frac{16}{\sqrt{3}}M_2$
$Z\langle\frac{1}{2},\mathrm{III}\rangle+$	$-\frac{8}{3\sqrt{2}}(M_2-M_1-3J)$	0	0	$-\frac{16}{\sqrt{3}}M_1$
$\widetilde{Z}\langle\frac{1}{2},\mathrm{III}\rangle$	0	$-\frac{16}{3\sqrt{2}}(M_2-M_1+3J)$	$-\frac{16}{\sqrt{3}}M_2$	0

Columns eight to eleven:

	$Z\langle\frac{1}{2},\mathrm{II}\rangle$	$\widetilde{Z}\langle\frac{1}{2},\mathrm{II}\rangle$	$Z\langle\frac{1}{2},\mathrm{III}\rangle$	$\widetilde{Z}\langle\frac{1}{2},\mathrm{III}\rangle$
$Z[0,\mathrm{I}]$	0	0	0	0
$Z[0,\mathrm{II}]$	0	0	0	0
$Z[1,\mathrm{I}]$	0	0	0	0
$Z\langle\frac{1}{2},\mathrm{I}\rangle$	$\frac{32}{3\sqrt{2}}(M_2-M_1-3J)$	0	$\frac{32}{3\sqrt{2}}(M_2-M_1+3J)$	0
$\widetilde{Z}\langle\frac{1}{2},\mathrm{I}\rangle$	0	$\frac{32}{3\sqrt{2}}(M_2-M_1+3J)$	0	$\frac{32}{3\sqrt{2}}(M_2-M_1-3J)$
$Z\langle 0,\mathrm{II}\rangle$	$-\frac{32}{\sqrt{3}}(M_2+2)$	0	0	$-\frac{32}{\sqrt{3}}(M_1+2)$
$\widetilde{Z}\langle 0,\mathrm{II}\rangle$	0	$-\frac{32}{\sqrt{3}}(M_1+2)$	$-\frac{32}{\sqrt{3}}(M_2+2)$	0
$Z\langle\frac{1}{2},\mathrm{II}\rangle$	H_0	0	0	0
$\widetilde{Z}\langle\frac{1}{2},\mathrm{II}\rangle$	0	H_0	0	0
$Z\langle\frac{1}{2},\mathrm{III}\rangle$	0	0	$H_0-\frac{64}{3}(M_2-M_1)$	0
$\widetilde{Z}\langle\frac{1}{2},\mathrm{III}\rangle$	0	0	0	$H_0+\frac{64}{3}(M_2-M_1)$

This matrix has the following eigenvalues:

$$\lambda_1 \;=\; H_0 + \tfrac{32}{3}(M_2 - M_1)\,,$$

$$
\begin{array}{rcl}
\lambda_2 &=& H_0 - \frac{32}{3}(M_2 - M_1)\,,\\
\lambda_3 &=& H_0\,,\\
\lambda_4 &=& H_0 + 24 + 4\sqrt{H_0+36}\,,\\
\lambda_5 &=& H_0 + 24 - 4\sqrt{H_0+36}\,,\\
\lambda_6 &=& H_0 + \frac{32}{3}(M_2 - M_1) + 16 + 4\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16}\,,\\
\lambda_7 &=& H_0 + \frac{32}{3}(M_2 - M_1) + 16 - 4\sqrt{H_0 + \frac{32}{3}(M_2 - M_1) + 16}\,,\\
\lambda_8 &=& H_0 - \frac{32}{3}(M_2 - M_1) + 16 + 4\sqrt{H_0 - \frac{32}{3}(M_2 - M_1) + 16}\,,\\
\lambda_9 &=& H_0 - \frac{32}{3}(M_2 - M_1) + 16 - 4\sqrt{H_0 - \frac{32}{3}(M_2 - M_1) + 16}\,,\\
\lambda_{10} &=& \lambda_{11} = H_0 + 32\,.
\end{array}
\tag{3.4.91}
$$

The eigenvalues $\lambda_1, \lambda_2, \lambda_4, \lambda_5$, equal to the one–form physical ones, are the longitudinal eigenvalues. The other seven are the physical two-form eigenvalues.

As in the case of the one–form, by removing rows and columns we find the matrix of each exceptional series, and the corresponding eigenvalues:

A_R	$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}$
A_1	$\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11}$
A_1^*	$\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}$
A_2	$\lambda_1, \lambda_6, \lambda_7$
A_2^*	$\lambda_2, \lambda_8, \lambda_9$
A_3	$\lambda_1, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_{10}, \lambda_{11}$
A_3^*	$\lambda_2, \lambda_4, \lambda_5, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}$
A_4	$\lambda_1, \lambda_4, \lambda_6, \lambda_7, \lambda_{10}$
A_4^*	$\lambda_2, \lambda_4, \lambda_8, \lambda_9, \lambda_{10}$
A_5	λ_1, λ_6
A_5^*	λ_2, λ_8
A_6	$\lambda_3, \lambda_4, \lambda_5, \lambda_{10}, \lambda_{11}$
A_7	$\lambda_4, \lambda_5, \lambda_{10}, \lambda_{11}$
A_8	λ_3, λ_4

(3.4.92)

The two–form operator matrix in the representations $^{++}$ is the following 5×5 matrix:
Columns one to two:

$M_{(1)^2(0)}$	$Z\langle 0,\mathrm{I}\rangle$	$Z\langle \frac{1}{2},\mathrm{I}\rangle$
$Z\langle 0,\mathrm{I}\rangle$	$H_0 - \frac{32}{3}(M_2 - M_1)$	$\frac{16}{\sqrt{3}}(M_1+2)$
$Z\langle \frac{1}{2},\mathrm{I}\rangle$	$\frac{32}{\sqrt{3}}M_2$	$H_0 + 32$
$Z\langle \frac{1}{2},\mathrm{II}\rangle$	0	$-\frac{16}{3\sqrt{2}}(M_2 - M_1 + 3J - 3)$
$Z\langle \frac{1}{2},\mathrm{III}\rangle$	0	$-\frac{16}{3\sqrt{2}}(M_2 - M_1 - 3J - 3)$
$Z\langle 1,\mathrm{I}\rangle$	0	$\frac{32}{\sqrt{3}}(M_2 - 1)$

Columns three to five:

$M_{(1)^2(0)}$	$Z\langle \frac{1}{2},\mathrm{II}\rangle$	$Z\langle \frac{1}{2},\mathrm{III}\rangle$	$Z\langle 1,\mathrm{I}\rangle$
$Z\langle 0,\mathrm{I}\rangle$	0	0	0
$Z\langle \frac{1}{2},\mathrm{I}\rangle$	$\frac{32}{3\sqrt{2}}(M_2 - M_1 - 3J - 6)$	$\frac{32}{3\sqrt{2}}(M_2 - M_1 + 3J)$	$\frac{16}{\sqrt{3}}(M_1+3)$
$Z\langle \frac{1}{2},\mathrm{II}\rangle$	$H_0 + \frac{32}{3}(M_2 - M_1) - 32$	0	0
$Z\langle \frac{1}{2},\mathrm{III}\rangle$	0	$H_0 - \frac{32}{3}(M_2 - M_1) + 32$	0
$Z\langle 1,\mathrm{I}\rangle$	0	0	$H_0 + \frac{32}{3}(M_2 - M_1) - 32$

It has eigenvalues

$$\begin{aligned}
\lambda_1 &= H_0 + \tfrac{32}{3}(M_2 - M_1)\,, \\
\lambda_2 &= H_0 + \tfrac{32}{3}(M_2 - M_1) + 16 + 4\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}\,, \\
\lambda_3 &= H_0 + \tfrac{32}{3}(M_2 - M_1) + 16 - 4\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}\,, \\
\lambda_4 &= H_0 + 32\,, \\
\lambda_5 &= H_0 + \tfrac{64}{3}(M_2 - M_1) - 32\,.
\end{aligned} \tag{3.4.93}$$

The eigenvalue λ_1, equal to the physical eigenvalue of the $(^{++})$ one–form, is longitudinal. The other four are the physical eigenvalues.

B_R	$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$
B_1	$\lambda_1, \lambda_2, \lambda_3, \lambda_5$
B_2	λ_5
B_3	$\lambda_1, \lambda_2, \lambda_3$
B_4	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$
B_5	λ_4
B_6	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$
B_7	$\lambda_1, \lambda_2, \lambda_4$
B_8	λ_4
B_9	λ_4
B_{10}	λ_4

(3.4.94)

For the $^{--}$ representations, the eigenvalues are the conjugates ($M_2 \leftrightarrow M_1$) of the ones in (3.4.94).

We can use the eigenvalues of the two–form harmonic to determine the masses of the AdS_4 vector field Z.

The three-form

The H decomposition of the three–form in H–irreducible fragments has been done in [8]:

$$\begin{aligned}
\mathcal{Y}^{ABC} &= \varepsilon^{ABCD}\left\{\lambda^D_{3i}\langle 3|\mathrm{I}\rangle_i + \lambda^D_{i3}\langle 3|\mathrm{I}\rangle^*_i\right\}\,, \\
\mathcal{Y}^{ABm} &= \lambda^A_{i3}\lambda^B_{j3}\varepsilon^{ij}\{\sigma^m_{21}\langle 3|\mathrm{II}\rangle_. + \sigma^m_{12}\langle 3|\mathrm{III}\rangle_.\} + \lambda^A_{3i}\lambda^B_{3j}\varepsilon^{ij}\{\sigma^m_{12}\langle 3|\mathrm{II}\rangle^*_. + \sigma^m_{21}\langle 3|\mathrm{III}\rangle^*_.\} + \\
&\quad + i\lambda^{[A}_{i3}\lambda^{B]}_{3i}\{\sigma^m_{21}\langle 3|\mathrm{IV}\rangle_. + \sigma^m_{12}\langle 3|\mathrm{IV}\rangle^*_.\} + \lambda^{[A}_{i3}\lambda^{B]}_{3j}\{\sigma^m_{21}\varepsilon^{ik}\langle 3|\mathrm{I}\rangle_{kj} - \sigma^m_{12}\varepsilon^{jk}\langle 3|\mathrm{I}\rangle^*_{ik}\}\,, \\
\mathcal{Y}^{AB3} &= \lambda^A_{3i}\lambda^B_{3j}\varepsilon^{ij}\langle 3|\mathrm{I}\rangle_. + i\lambda^A_{i3}\lambda^B_{j3}\varepsilon^{ij}\langle 3|\mathrm{I}\rangle^*_. + i\lambda^{[A}_{i3}\lambda^{B]}_{3i}[3|\mathrm{I}]_. + \lambda^{[A}_{i3}\lambda^{B]}_{3j}\varepsilon^{ik}[3|\mathrm{I}]_{kj}\,, \\
\mathcal{Y}^{Amn} &= \varepsilon^{mn}\{\lambda^A_{3i}\langle 3|\mathrm{II}\rangle_i + \lambda^A_{i3}\langle 3|\mathrm{II}\rangle^*_i\}\,, \\
\mathcal{Y}^{Am3} &= \lambda^A_{3i}\{\sigma^m_{12}\langle 3|\mathrm{IV}\rangle_i + \sigma^m_{21}\langle 3|\mathrm{III}\rangle_i\} + \lambda^A_{i3}\{\sigma^m_{21}\langle 3|\mathrm{IV}\rangle^*_i + \sigma^m_{12}\langle 3|\mathrm{III}\rangle^*_i\}\,, \\
\mathcal{Y}^{mn3} &= \varepsilon^{mn}[3|\mathrm{II}]_.\,,
\end{aligned}$$

where the fragments of type ^{0} are:

$$\begin{aligned}
\langle 3|\mathrm{I}\rangle_{ij} &= \mathcal{H}^{[1,-2i,-3i](c)}_{ij}\cdot\pi\langle 1,\mathrm{I}\rangle\,, \\
\langle 3|\mathrm{I}\rangle^*_{ij} &= -\varepsilon^{ik}\varepsilon^{jl}\mathcal{H}^{[1,2i,3i](c)}_{kl}\cdot\widetilde{\pi}\langle 1,\mathrm{I}\rangle\,, \\
[3|\mathrm{I}]_{ij} &= \mathcal{H}^{[1,0,0](c)}_{ij}\cdot\pi[1,\mathrm{I}]\,, \\
\langle 3|\mathrm{I}\rangle_i &= \mathcal{H}^{[1/2,3i,-3i/2](a)}_i\cdot\pi\langle\tfrac{1}{2},\mathrm{I}\rangle\,,
\end{aligned}$$

$$
\begin{array}{rcl}
\langle 3|\mathrm{I}\rangle_i^* &=& \varepsilon^{ij}\mathcal{H}_j^{[1/2,-3i,3i/2](b)} \cdot \widetilde{\pi}\langle \tfrac{1}{2}, \mathrm{I}\rangle \,, \\
\langle 3|\mathrm{II}\rangle_i &=& \mathcal{H}_i^{[1/2,3i,-3i/2](a)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{II}\rangle \,, \\
\langle 3|\mathrm{II}\rangle_i^* &=& \varepsilon^{ij}\mathcal{H}_j^{[1/2,-3i,3i/2](b)} \cdot \widetilde{\pi}\langle \tfrac{1}{2}, \mathrm{II}\rangle \,, \\
\langle 3|\mathrm{III}\rangle_i &=& \mathcal{H}_i^{[1/2,i,-9i/2](a)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{III}\rangle \,, \\
\langle 3|\mathrm{III}\rangle_i^* &=& \varepsilon^{ij}\mathcal{H}_j^{[1/2,-i,9i/2](b)} \cdot \widetilde{\pi}\langle \tfrac{1}{2}, \mathrm{III}\rangle \,, \\
\langle 3|\mathrm{IV}\rangle_i &=& \mathcal{H}_i^{[1/2,5i,3i/2](a)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{IV}\rangle \,, \\
\langle 3|\mathrm{IV}\rangle_i^* &=& \varepsilon^{ij}\mathcal{H}_j^{[1/2,-5i,-3i/2](b)} \cdot \widetilde{\pi}\langle \tfrac{1}{2}, \mathrm{IV}\rangle \,, \\
\langle 3|\mathrm{IV}\rangle_. &=& \mathcal{H}^{[0,-2i,-3i]} \cdot \pi\langle 0, \mathrm{IV}\rangle \,, \\
\langle 3|\mathrm{IV}\rangle_.^* &=& \mathcal{H}^{[0,2i,3i]} \cdot \widetilde{\pi}\langle 0, \mathrm{IV}\rangle \,, \\
[3|\mathrm{I}]_. &=& \mathcal{H}^{[0,0,0]} \cdot \pi[0, \mathrm{I}] \,, \\
[3|\mathrm{II}]_. &=& \mathcal{H}^{[0,0,0]} \cdot \pi[0, \mathrm{II}] \,,
\end{array}
$$

while the fragments of type $^{++}$ are:

$$
\begin{array}{rcl}
\langle 3|\mathrm{I}\rangle_{ij} &=& \mathcal{H}_{ij}^{[1,-2i,-3i](d)} \cdot \pi\langle 1, \mathrm{I}\rangle \,, \\
\langle 3|\mathrm{I}\rangle_{ij}^* &=& \varepsilon^{ik}\varepsilon^{jl}\mathcal{H}_{kl}^{[1,2i,3i](d)} \cdot \widetilde{\pi}\langle 1, \mathrm{I}\rangle \,, \\
[3|\mathrm{I}]_{ij} &=& \mathcal{H}_{ij}^{[1,0,0](d)} \cdot \pi[1, \mathrm{I}] \,, \\
\langle 3|\mathrm{I}\rangle_i &=& \mathcal{H}_i^{[1/2,3i,-3i/2](b)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{I}\rangle \,, \\
\langle 3|\mathrm{II}\rangle_i &=& \mathcal{H}_i^{[1/2,3i,-3i/2](b)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{II}\rangle \,, \\
\langle 3|\mathrm{III}\rangle_i &=& \mathcal{H}_i^{[1/2,i,-9i/2](b)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{III}\rangle \,, \\
\langle 3|\mathrm{IV}\rangle_i &=& \mathcal{H}_i^{[1/2,5i,3i/2](b)} \cdot \pi\langle \tfrac{1}{2}, \mathrm{IV}\rangle \,, \\
\langle 3|\mathrm{I}\rangle_. &=& \mathcal{H}^{[0,6i,-3i]} \cdot \pi\langle 0, \mathrm{I}\rangle \,, \\
\langle 3|\mathrm{II}\rangle_.^* &=& \mathcal{H}^{[0,8i,0]} \cdot \widetilde{\pi}\langle 0, \mathrm{II}\rangle \,, \\
\langle 3|\mathrm{III}\rangle_.^* &=& \mathcal{H}^{[0,4i,-6i]} \cdot \widetilde{\pi}\langle 0, \mathrm{III}\rangle \,.
\end{array}
$$

The fragments that are present in the type $^{--}$ series are the complex conjugates of the fragments above. The Laplace Beltrami operator for the transverse three-form $\mathcal{Y}^{[abc]}$, is a first-order differential operator, given by

$$
\begin{aligned}
\boxtimes^{[111]}\mathcal{Y}^{[abc]} \equiv M_{(1)^3}\mathcal{Y}^{[abc]} = \tfrac{1}{24}\epsilon^{abcd}{}_{mnr}\mathcal{D}_d\mathcal{Y}^{mnr} = \\
= \tfrac{1}{24}\epsilon^{abgd}{}_{mnr}\left[\mathcal{D}_d^H\mathcal{Y}^{mnr} + (\mathbb{M}_d)^m{}_s\mathcal{Y}^{snr} + (\mathbb{M}_d)^n{}_s\mathcal{Y}^{msr} + (\mathbb{M}_d)^r{}_s\mathcal{Y}^{mns}\right].
\end{aligned}
\tag{3.4.95}
$$

For the regular series of type ^{0} this operator acts on the AdS_4 fields as a 15×15 matrix:
Columns one to five:

$M_{(1)^3}$	$\pi\langle 1,\mathrm{I}\rangle$	$\widetilde{\pi}\langle 1,\mathrm{I}\rangle$	$\pi[1,\mathrm{I}]$	$\pi\langle \frac{1}{2},\mathrm{I}\rangle$	$\widetilde{\pi}\langle \frac{1}{2},\mathrm{I}\rangle$
$\pi\langle 1,\mathrm{I}\rangle$	Y	0	$\frac{-(2\,J+Y)}{2\,\sqrt{2}}$	0	0
$\widetilde{\pi}\langle 1,\mathrm{I}\rangle$	0	$-Y$	$\frac{-2\,J+Y}{2\,\sqrt{2}}$	0	0
$\pi[1,\mathrm{I}]$	$\frac{-2-2\,J+Y}{\sqrt{2}}$	$-\frac{2+2\,J+Y}{\sqrt{2}}$	1	0	0
$\pi\langle \frac{1}{2},\mathrm{I}\rangle$	0	0	0	0	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{I}\rangle$	0	0	0	0	0
$\pi\langle \frac{1}{2},\mathrm{II}\rangle$	0	0	$\frac{\frac{i}{2}\,(2+M_1)}{\sqrt{3}}$	$-i\,Y$	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{II}\rangle$	0	0	$\frac{\frac{i}{2}\,(2+M_2)}{\sqrt{3}}$	0	$-i\,Y$
$\pi\langle \frac{1}{2},\mathrm{III}\rangle$	$\frac{2+M_1}{2\,\sqrt{3}}$	0	0	$\frac{-(2\,J+Y)}{2\,\sqrt{2}}$	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{III}\rangle$	0	$\frac{2+M_2}{2\,\sqrt{3}}$	0	0	$\frac{2\,J-Y}{2\,\sqrt{2}}$
$\pi\langle \frac{1}{2},\mathrm{IV}\rangle$	0	$\frac{2+M_1}{2\,\sqrt{3}}$	0	$\frac{2\,J-Y}{2\,\sqrt{2}}$	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{IV}\rangle$	$\frac{2+M_2}{2\,\sqrt{3}}$	0	0	0	$\frac{-(2\,J+Y)}{2\,\sqrt{2}}$
$\pi\langle 0,\mathrm{IV}\rangle$	0	0	0	0	0
$\widetilde{\pi}\langle 0,\mathrm{IV}\rangle$	0	0	0	0	0
$\pi[0,\mathrm{I}\vert\rho]$	0	0	0	0	0
$\pi[0,\mathrm{II}\vert\rho]$	0	0	0	$\frac{2\,i\,(2+M_2)}{\sqrt{3}}$	$\frac{-2\,i\,(2+M_1)}{\sqrt{3}}$

(3.4.96)

Columns six to ten:

	$\pi\langle \frac{1}{2},\mathrm{II}\rangle$	$\widetilde{\pi}\langle \frac{1}{2},\mathrm{II}\rangle$	$\pi\langle \frac{1}{2},\mathrm{III}\rangle$	$\widetilde{\pi}\langle \frac{1}{2},\mathrm{III}\rangle$	$\pi\langle \frac{1}{2},\mathrm{IV}\rangle$
$\pi\langle 1,\mathrm{I}\rangle$	0	0	$\frac{2\,M_2}{\sqrt{3}}$	0	0
$\widetilde{\pi}\langle 1,\mathrm{I}\rangle$	0	0	0	$\frac{2\,M_1}{\sqrt{3}}$	$\frac{2\,M_2}{\sqrt{3}}$
$\pi[1,\mathrm{I}]$	$\frac{-2\,i\,M_2}{\sqrt{3}}$	$\frac{-2\,i\,M_1}{\sqrt{3}}$	0	0	0
$\pi\langle \frac{1}{2},\mathrm{I}\rangle$	$i\,Y$	0	$\frac{-2-2\,J+Y}{\sqrt{2}}$	0	$\frac{2+2\,J+Y}{\sqrt{2}}$
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{I}\rangle$	0	$i\,Y$	0	$\frac{2+2\,J+Y}{\sqrt{2}}$	0
$\pi\langle \frac{1}{2},\mathrm{II}\rangle$	0	0	0	0	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{II}\rangle$	0	0	0	0	0
$\pi\langle \frac{1}{2},\mathrm{III}\rangle$	0	0	1	0	0
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{III}\rangle$	0	0	0	1	0
$\pi\langle \frac{1}{2},\mathrm{IV}\rangle$	0	0	0	0	-1
$\widetilde{\pi}\langle \frac{1}{2},\mathrm{IV}\rangle$	0	0	0	0	0
$\pi\langle 0,\mathrm{IV}\rangle$	0	0	$\frac{-i\,(2+M_2)}{\sqrt{3}}$	0	0
$\widetilde{\pi}\langle 0,\mathrm{IV}\rangle$	0	0	0	$\frac{i\,(2+M_1)}{\sqrt{3}}$	$\frac{i\,(2+M_2)}{\sqrt{3}}$
$\pi[0,\mathrm{I}]$	$-\frac{2+M_2}{\sqrt{3}}$	$-\frac{2+M_1}{\sqrt{3}}$	0	0	0
$\pi[0,\mathrm{II}]$	0	0	0	0	0

(3.4.97)

Columns eleven to fifteen:

	$\widetilde{\pi}\langle\frac{1}{2},\mathrm{IV}\rangle$	$\pi\langle 0,\mathrm{IV}\rangle$	$\widetilde{\pi}\langle 0,\mathrm{IV}\rangle$	$\pi[0,\mathrm{I}]$	$\pi[0,\mathrm{II}]$
$\pi\langle 1,\mathrm{I}\rangle$	$\frac{2\,M_1}{\sqrt{3}}$	0	0	0	0
$\widetilde{\pi}\langle 1,\mathrm{I}\rangle$	0	0	0	0	0
$\pi[1,\mathrm{I}]$	0	0	0	0	0
$\pi\langle\frac{1}{2},\mathrm{I}\rangle$	0	0	0	0	$\frac{-i\,M_1}{\sqrt{3}}$
$\widetilde{\pi}\langle\frac{1}{2},\mathrm{I}\rangle$	$\frac{-2-2\,J+Y}{\sqrt{2}}$	0	0	0	$\frac{i\,M_2}{\sqrt{3}}$
$\pi\langle\frac{1}{2},\mathrm{II}\rangle$	0	0	0	$-\frac{M_1}{\sqrt{3}}$	0
$\widetilde{\pi}\langle\frac{1}{2},\mathrm{II}\rangle$	0	0	0	$-\frac{M_2}{\sqrt{3}}$	0
$\pi\langle\frac{1}{2},\mathrm{III}\rangle$	0	$\frac{i\,M_1}{\sqrt{3}}$	0	0	0
$\widetilde{\pi}\langle\frac{1}{2},\mathrm{III}\rangle$	0	0	$\frac{-i\,M_2}{\sqrt{3}}$	0	0
$\pi\langle\frac{1}{2},\mathrm{IV}\rangle$	0	0	$\frac{-i\,M_1}{\sqrt{3}}$	0	0
$\widetilde{\pi}\langle\frac{1}{2},\mathrm{IV}\rangle$	-1	$\frac{i\,M_2}{\sqrt{3}}$	0	0	0
$\pi\langle 0,\mathrm{IV}\rangle$	$\frac{-i\,(2+M_1)}{\sqrt{3}}$	$-Y$	0	$\frac{2\,J+Y}{2\,\sqrt{2}}$	0
$\widetilde{\pi}\langle 0,\mathrm{IV}\rangle$	0	0	Y	$\frac{-2\,J+Y}{2\,\sqrt{2}}$	0
$\pi[0,\mathrm{I}]$	0	$\frac{2+2\,J-Y}{\sqrt{2}}$	$-\frac{2+2\,J+Y}{\sqrt{2}}$	-1	-1
$\pi[0,\mathrm{II}]$	0	0	0	-2	0

(3.4.98)

This matrix has the following eigenvalues:

$$
\begin{aligned}
\lambda_1 &= \tfrac{1}{4}\sqrt{H_0+\tfrac{32}{3}(M_2-M_1)+16}\,,\\
\lambda_2 &= \tfrac{1}{4}\sqrt{H_0-\tfrac{32}{3}(M_2-M_1)+16}\,,\\
\lambda_3 &= -\tfrac{1}{4}\sqrt{H_0+\tfrac{32}{3}(M_2-M_1)+16}\,,\\
\lambda_4 &= -\tfrac{1}{4}\sqrt{H_0-\tfrac{32}{3}(M_2-M_1)+16}\,,\\
\lambda_5 &= \tfrac{1}{4}\sqrt{H_0+36}-\tfrac{1}{2}\,,\\
\lambda_6 &= -\tfrac{1}{4}\sqrt{H_0+36}-\tfrac{1}{2}\,,\\
\lambda_7 &= -\tfrac{1}{4}\sqrt{H_0+4}+\tfrac{1}{2}\,,\\
\lambda_8 &= \tfrac{1}{4}\sqrt{H_0+4}+\tfrac{1}{2}\,,\\
\lambda_9 &= \ldots=\lambda_{15}=0\,.
\end{aligned}
\tag{3.4.99}
$$

We note that seven eigenvalues are 0. They correspond to the longitudinal three-forms ($\mathcal{Y}^{(3)}=\mathcal{D}\wedge\mathcal{Y}^{(2)}$), which are annihilated by $\boxtimes^{[111]}$ ($={}^*\mathcal{D}\wedge$).

As in the cases of the one–form and of the two–form, by removing rows and columns

we find the matrix for each exceptional series, and the corresponding eigenvalues:

A_R	$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$
A_1	$\lambda_1, \lambda_3, \lambda_5, \lambda_6, \lambda_7, \lambda_8$
A_1^*	$\lambda_2, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8$
A_2	λ_1, λ_3
A_2^*	λ_2, λ_4
A_3	$\lambda_1, \lambda_3, \lambda_5, \lambda_6$
A_3^*	$\lambda_2, \lambda_4, \lambda_5, \lambda_6$
A_4	$\lambda_1, \lambda_3, \lambda_6$
A_4^*	$\lambda_2, \lambda_4, \lambda_6$
A_5	λ_1, λ_3
A_5^*	λ_2, λ_4
A_6	$\lambda_5, \lambda_6, \lambda_7, \lambda_8$
A_7	λ_5, λ_6
A_8	λ_6, λ_8

(3.4.100)

The two–form operator matrix for the regular series of type $^{++}$ is the following 10×10 matrix:
Columns one to five:

	$\pi\langle 1, \mathrm{I}\rangle$	$\widetilde{\pi}\langle 1, \mathrm{I}\rangle$	$\pi[1, \mathrm{I}]$	$\pi\langle \frac{1}{2}, \mathrm{I}\rangle$	$\pi\langle \frac{1}{2}, \mathrm{II}\rangle$
$\pi\langle 1, \mathrm{I}\rangle$	Y	0	$\frac{-(2\,J+Y)}{2\,\sqrt{2}}$	0	0
$\widetilde{\pi}\langle 1, \mathrm{I}\rangle$	0	$-Y$	$\frac{-(-2\,J+Y)}{2\,\sqrt{2}}$	0	0
$\pi[1, \mathrm{I}]$	$\frac{-2-2\,J+Y}{\sqrt{2}}$	$\frac{2+2\,J+Y}{\sqrt{2}}$	1	0	$\frac{-2\,i\,(-1+M_2)}{\sqrt{3}}$
$\pi\langle \frac{1}{2}, \mathrm{I}\rangle$	0	0	0	0	$i\,Y$
$\pi\langle \frac{1}{2}, \mathrm{II}\rangle$	0	0	$\frac{i\,(3+M_1)}{\sqrt{3}}$	$-i\,Y$	0
$\pi\langle \frac{1}{2}, \mathrm{III}\rangle$	$\frac{3+M_1}{\sqrt{3}}$	0	0	$\frac{-(2\,J+Y)}{2\,\sqrt{2}}$	0
$\pi\langle \frac{1}{2}, \mathrm{IV}\rangle$	0	$-\frac{3+M_1}{\sqrt{3}}$	0	$\frac{2\,J-Y}{2\,\sqrt{2}}$	0
$\pi\langle 0, \mathrm{I}\rangle$	0	0	0	0	$\frac{i\,(2+M_1)}{\sqrt{3}}$
$\widetilde{\pi}\langle 0, \mathrm{II}\rangle$	0	0	0	0	0
$\widetilde{\pi}\langle 0, \mathrm{III}\rangle$	0	0	0	0	0

(3.4.101)

Columns six to ten:

	$\pi\langle \frac{1}{2}, \mathrm{III}\rangle$	$\pi\langle \frac{1}{2}, \mathrm{IV}\rangle$	$\pi\langle 0, \mathrm{I}\rangle$	$\widetilde{\pi}\langle 0, \mathrm{II}\rangle$	$\widetilde{\pi}\langle 0, \mathrm{III}\rangle$
$\pi\langle 1, \mathrm{I}\rangle$	$\frac{2\,(-1+M_2)}{\sqrt{3}}$	0	0	0	0
$\widetilde{\pi\langle 1, \mathrm{I}\rangle}$	0	$\frac{-2\,(-1+M_2)}{\sqrt{3}}$	0	0	0
$\pi[1, \mathrm{I}]$	0	0	0	0	0
$\pi\langle \frac{1}{2}, \mathrm{I}\rangle$	$\frac{-2-2\,J+Y}{\sqrt{2}}$	$\frac{2+2\,J+Y}{\sqrt{2}}$	0	0	0
$\pi\langle \frac{1}{2}, \mathrm{II}\rangle$	0	0	$\frac{-2\,i\,M_2}{\sqrt{3}}$	0	0
$\pi\langle \frac{1}{2}, \mathrm{III}\rangle$	1	0	0	0	$\frac{-2\,M_2}{\sqrt{3}}$
$\pi\langle \frac{1}{2}, \mathrm{IV}\rangle$	0	-1	0	$\frac{2\,M_2}{\sqrt{3}}$	0
$\pi\langle 0, \mathrm{I}\rangle$	0	0	-1	$-\frac{2+2\,J+Y}{\sqrt{2}}$	$\frac{2+2\,J-Y}{\sqrt{2}}$
$\widetilde{\pi}\langle 0, \mathrm{II}\rangle$	0	$\frac{2+M_1}{\sqrt{3}}$	$\frac{-2\,J+Y}{2\,\sqrt{2}}$	Y	0
$\widetilde{\pi}\langle 0, \mathrm{III}\rangle$	$-\frac{2+M_1}{\sqrt{3}}$	0	$\frac{2\,J+Y}{2\,\sqrt{2}}$	0	$-Y$

(3.4.102)

It has eigenvalues:

$$\begin{aligned}
\lambda_1 &= \tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}\,, \\
\lambda_2 &= -\tfrac{1}{4}\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}\,,
\end{aligned}$$

$$
\begin{aligned}
\lambda_3 &= \tfrac{1}{4}\sqrt{H_0+36}-\tfrac{1}{2}\,, \\
\lambda_4 &= -\tfrac{1}{4}\sqrt{H_0+36}-\tfrac{1}{2}\,, \\
\lambda_5 &= -\tfrac{1}{4}\sqrt{H_0+\tfrac{64}{3}(M_2-M_1)-28}+\tfrac{1}{2}\,, \\
\lambda_6 &= \tfrac{1}{4}\sqrt{H_0+\tfrac{64}{3}(M_2-M_1)-28}+\tfrac{1}{2}\,, \\
\lambda_7 &= \ldots = \lambda_{10} = 0\,.
\end{aligned}
\tag{3.4.103}
$$

The complete table of eigenvalues for the type $^{++}$ series is:

B_R	$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$
B_1	$\lambda_1, \lambda_2, \lambda_5, \lambda_6$
B_2	λ_5, λ_6
B_3	λ_1, λ_2
B_4	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$
B_5	λ_3, λ_4
B_6	$\lambda_1, \lambda_2, \lambda_3, \lambda_4$
B_7	$\lambda_2, \lambda_3, \lambda_4$
B_8	λ_4
B_9	λ_3, λ_4
B_{10}	λ_4
B_{11}	λ_4

(3.4.104)

For the representations of the $^{--}$ series, the eigenvalues are the conjugates of the ones in (3.4.104).

The spinor

The harmonic analysis of the eight–component Majorana spinor has been completely worked out in [21]. We reformulate these results in our framework, in order to facilitate the matching of the spectrum with the $\mathcal{N}=2$ multiplets.

The decomposition of the spinor in its H–irreducible components is

$$
\eta = \begin{pmatrix} \langle\frac{1}{2}|\mathrm{I}\rangle_i \\ \langle\frac{1}{2}|\mathrm{I}\rangle_{\cdot} \\ \langle\frac{1}{2}|\mathrm{II}\rangle_{\cdot} \\ -i\sigma_2\langle\frac{1}{2}|\mathrm{I}\rangle_i^* \\ \langle\frac{1}{2}|\mathrm{II}\rangle_{\cdot}^* \\ -\langle\frac{1}{2}|\mathrm{I}\rangle_{\cdot}^* \end{pmatrix}
\tag{3.4.105}
$$

where

$$
\begin{aligned}
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i &= \mathcal{H}_i^{[1/2,-i,-3i/2]\xi}\cdot\chi\langle\tfrac{1}{2},\mathrm{I}\rangle\,, \\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_{\cdot} &= \mathcal{H}^{[0,2i,-3i]}\cdot\chi\langle 0,\mathrm{I}\rangle\,, \\
\langle\tfrac{1}{2}|\mathrm{II}\rangle_{\cdot} &= \mathcal{H}^{[0,-4i,0]}\cdot\chi\langle 0,\mathrm{II}\rangle\,, \\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i^* &= \pm\varepsilon^{ij}\mathcal{H}_j^{[1/2,i,3i/2]\xi}\cdot\widetilde{\chi}\langle\tfrac{1}{2},\mathrm{I}\rangle\,, \\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_{\cdot}^* &= \mathcal{H}^{[0,-2i,3i]}\cdot\widetilde{\chi}\langle 0,\mathrm{I}\rangle\,, \\
\langle\tfrac{1}{2}|\mathrm{II}\rangle_{\cdot}^* &= \mathcal{H}^{[0,-4i,0]}\cdot\widetilde{\chi}\langle 0,\mathrm{II}\rangle.
\end{aligned}
\tag{3.4.106}
$$

The fragments of type $^+$ are

$$\begin{aligned}
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i &= \mathcal{H}_i^{[1/2,-i,-3i/2](b)}\cdot\chi\langle\tfrac{1}{2},\mathrm{I}\rangle\,,\\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_\cdot &= \mathcal{H}^{[0,2i,-3i]}\cdot\chi\langle 0,\mathrm{I}\rangle\,,\\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i^* &= \varepsilon^{ij}\mathcal{H}_j^{[1/2,i,3i/2](b)}\cdot\widetilde{\chi}\langle\tfrac{1}{2},\mathrm{I}\rangle\,,\\
\langle\tfrac{1}{2}|\mathrm{II}\rangle_\cdot^* &= \mathcal{H}^{[0,4i,0]}\cdot\widetilde{\chi}\langle 0,\mathrm{II}\rangle\,.
\end{aligned}\tag{3.4.107}$$

For the regular series $^+$ the spinor operator acts on the AdS_4 fields as a 4×4 matrix, whose eigenvalues are:

$$\begin{aligned}
\lambda_1 &= -6+\sqrt{H_0+36}\,,\\
\lambda_2 &= -6-\sqrt{H_0+36}\,,\\
\lambda_3 &= -8+\sqrt{H_0+16+\tfrac{32}{3}(M_2-M_1)}\,,\\
\lambda_4 &= -8-\sqrt{H_0+16+\tfrac{32}{3}(M_2-M_1)}\,.
\end{aligned}\tag{3.4.108}$$

The eigenvalues for each exceptional series are

$A_R^+, A_1^+, A_3^+, A_4^+$	$\lambda_1,\lambda_2,\lambda_3,\lambda_4$
A_2^+, A_5^+	λ_3,λ_4
A_1^{+*}, A_6^+	λ_1,λ_2
A_3^{+*}, A_7^+	λ_1,λ_2
A_4^{+*}, A_8^+	λ_1

(3.4.109)

The fragments of type $^-$ are

$$\begin{aligned}
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i &= \mathcal{H}_i^{[1/2,-i,-3i/2](a)}\cdot\chi\langle\tfrac{1}{2},\mathrm{I}\rangle\,,\\
\langle\tfrac{1}{2}|\mathrm{II}\rangle_\cdot &= \mathcal{H}^{[0,-4i,0]}\cdot\chi\langle 0,\mathrm{II}\rangle\,,\\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_i^* &= \varepsilon^{ij}\mathcal{H}_j^{[1/2,i,3i/2](a)}\cdot\widetilde{\chi\langle\tfrac{1}{2},\mathrm{I}\rangle}\,,\\
\langle\tfrac{1}{2}|\mathrm{I}\rangle_\cdot^* &= \mathcal{H}^{[0,-2i,3i]}\cdot\widetilde{\chi\langle 0,\mathrm{II}\rangle}\,.
\end{aligned}\tag{3.4.110}$$

For the regular series $^-$ the spinor operator acts on the AdS_4 fields as a 4×4 matrix, whose eigenvalues are:

$$\begin{aligned}
\lambda_1 &= -6+\sqrt{H_0+36}\,,\\
\lambda_2 &= -6-\sqrt{H_0+36}\,,\\
\lambda_3 &= -8+\sqrt{H_0+16-\tfrac{32}{3}(M_2-M_1)}\,,\\
\lambda_4 &= -8-\sqrt{H_0+16-\tfrac{32}{3}(M_2-M_1)}\,.
\end{aligned}\tag{3.4.111}$$

The eigenvalues for each exceptional series are:

$A_R^-, A_1^{-*}, A_3^{-*}, A_4^{-*}$	$\lambda_1,\lambda_2,\lambda_3,\lambda_4$
A_2^{-*}, A_5^{-*}	λ_3,λ_4
A_1^-, A_6^-	λ_1,λ_2
A_3^-, A_7^-	λ_1,λ_2
A_4^-, A_8^-	λ_1

(3.4.112)

3.4.6 Matching the spectrum with the $Osp(2|4)$ multiplets

As already mentioned, the structures of the long multiplets that arise from $\mathcal{N}=2$ compactifications of eleven–dimensional supergravity in AdS_4 has been found in [23]. The structure and the G' representations of the long graviton, the long gravitino and the massless multiplets are known since the eighties [23]. The structure of the long vector multiplet can be very easily derived, as shown in chapter 2. However this is not the case for the the short multiplets: the method of norms become very cumbersome after the B_2 sector. So we have joined our knowledge on long multiplets and on the part of short multiplets arising from the simpler norm calculation, which are shown in chapter 2, and information arising from harmonic analysis, in order to get two results at the same time:

1. filling the blanks in the structure of $\mathcal{N}=2$ short multiplets, verifying which fields disappear, and if there are new shortening conditions;
2. finding the complete spectrum of this supergravity, even the part of the spectrum which has not been directly found from harmonic analysis.

In this section I rewrite the tables of the $\mathcal{N}=2$ supermultiplets, already given in chapter two, because I have to assign to each field its name following the definitions given in section 22. This is necessary in order to follow the reasoning of filling the multiplets with these fields. In each of these tables, the fields whose presence in the corresponding multiplet can be established by means of the norm evaluation discussed in chapter 2 are denoted by an asterisk, in order to distinguish them from the fields whose presence is established by the discussion below, utilizing the harmonic analysis results and the $\mathcal{N}=2 \longrightarrow \mathcal{N}=1$ decomposition. The fields in the long and massless multiplets have all the asterisk because as I said the structure of these multiplets was known before performing harmonic analysis.

We use a procedure of *exhaustion*, i.e. one starts with one of the four different types of multiplets for which all the masses of a certain field component are most easily retrieved (this is for instance the case for the graviton field of the graviton multiplet) and using the mass relations (3.1.71), (3.1.73), (3.1.72), one calculates all the masses of the other types of fields present in the multiplet. One uses also the information that all the fields in a multiplet are in the same irreducible $G' = SU(3) \times SU(2)$ representation and that their hypercharges are related according to the group theoretical structure of the multiplets shown in tables 3.7,…,3.15. So one knows in which G representation to find the other fields of the multiplet, whose masses have been determined. Then, upon using the relations (3.1.70), these masses are compared with the eigenvalues of the invariant operators on the spinor, the one–form, the two–form or the three–form depending on the type of field one is considering. The upshot of this is that some of these eigenvalues yield all the masses obtained from the mass relations. However, the remaining eigenvalues signal the existence of some extra masses which then pertain to other fields that are to be found in other multiplets. In this way one establishes the existence of new unknown multiplets and determines their structure by filling out their field content. After repeatedly applying this procedure one will have filled out all the existing multiplets in the spectrum.

I should remark here that we did not calculate the eigenvalues of the Lichnerowicz and Rarita–Schwinger operators $M_{(2)(0)^2}$ and $M_{(3/2)(1/2)^2}$. However we succeeded in finding the complete multiplet structure without making use of this. The AdS_4 fields whose spectrum is determined by $M_{(2)(0)^2}$ and $M_{(3/2)(1/2)^2}$ are the scalar field ϕ and the transverse spinor field λ_T (see (3.1.61)). We can fill the multiplets without knowing the spectrum of these

two fields with the help of the $\mathcal{N}=2 \to \mathcal{N}=1$ decompositions (2.4.66),…,(2.4.78). If we know every field of a multiplet except for ϕ and λ_T, we can deduce which ϕ and λ_T are present by trying to organize the $\mathcal{N}=2$ multiplet in $\mathcal{N}=1$ multiplets. There is no ambiguity, because no $\mathcal{N}=1$ multiplet is built using ϕ and λ_T fields alone. In particular, a Wess Zumino multiplet with one λ_T and two ϕ 's is not allowed, since it has to contain both a scalar and a pseudoscalar.

In practice one starts with the graviton multiplet since the masses of the graviton field in the different representations are immediate to derive, being the eigenvalues of the scalar operator $M_{(0)^3}$. By means of the above procedure, one *exhausts* all the spin–$\frac{3}{2}$ fields in the graviton multiplet comparing the masses of the spin–$\frac{3}{2}$ fields in the graviton multiplet with the eigenvalues of the operator $M_{(1/2)^3}$. The spin–$\frac{3}{2}$ fields that provide the remaining eigenvalues of the operator $M_{(1/2)^3}$, can only be the highest–spin component gravitino fields of the gravitino multiplet and hence we know all the masses of the gravitinos in the gravitino multiplet. At this stage we can repeat the same procedure. We use the eigenvalues of the one–form operator $M_{(1)(0)^2}$ to identify the vector fields A and W and we use the eigenvalues of the two–form operator $M_{(1)^2(0)}$ to identify the vector fields Z in the graviton and the gravitino multiplet. The remaining vector fields constitute the highest–component vector fields of the vector multiplet. Then we determine the masses of the longitudinal spinors, provided by the eigenvalues of the operator $M_{(1/2)^3}$, and we find the longitudinal spinors of the gravitino and vector multiplet. The remaining longitudinal spinors belong to hypermultiplets. At the end we determine the masses of the scalars S, Σ, that are provided by the eigenvalues of $M_{(0)^3}$, and of the pseudoscalar π, provided by the eigenvalues of the three–form operator $M_{(1)^3}$. At this point, the matching of the spectrum with the multiplets will be complete.

Since we are in particular interested in multiplet shortening, it is of utmost importance to pay attention to what happens with the eigenvalues in the exceptional series. As it is clear from tables (3.4.86), (3.4.100), (3.4.104) of the eigenvalues, there are always less eigenvalues present when the operators act on the harmonics in the exceptional series. This is reflected into the fact that certain field components are not present in the multiplets, thus multiplet shortening.

In the next sections I give a detailed discussion of the matching of the multiplets. Doing so I show that the information collected about the invariant operators on the zero form, the one–form, the two–form, the three–form and the spinor is in perfect agreement with the group theoretical information given in [23] and in chapter 2 of this thesis.

The graviton multiplet

As pointed out above, the graviton multiplet is the appropriate multiplet to start with. In particular we look at the spin–two graviton field. The mass of the graviton is given by the eigenvalue of the scalar operator (see eq.s (3.1.70)):

$$m_h^2 = M_{(0)^3} \equiv H_0 \,. \tag{3.4.113}$$

Using table 3.2 we find that its harmonics can sit in all the G representations of the series

$$A_R^0, A_1^0, A_1^{*0}, A_3^0, A_3^{*0}, A_4^0, A_4^{*0}, A_6^0, A_7^0, A_8^0 \,. \tag{3.4.114}$$

Remember that the superscripts ^{0} mean that the hypercharge is $Y = \frac{2}{3}(M_2 - M_1)$.

	Spin	Energy	Hypercharge	Mass (^{2})	Name
$*$	2	E_0+1	y_0	$16(E_0+1)(E_0-2)$	h
$*$	$\frac{3}{2}$	$E_0+\frac{3}{2}$	y_0-1	$-4E_0-4$	χ^-
$*$	$\frac{3}{2}$	$E_0+\frac{3}{2}$	y_0+1	$-4E_0-4$	χ^-
$*$	$\frac{3}{2}$	$E_0+\frac{1}{2}$	y_0-1	$4E_0-8$	χ^+
$*$	$\frac{3}{2}$	$E_0+\frac{1}{2}$	y_0+1	$4E_0-8$	χ^+
$*$	1	E_0+2	y_0	$16E_0(E_0+1)$	W
$*$	1	E_0+1	y_0-2	$16E_0(E_0-1)$	Z
$*$	1	E_0+1	y_0+2	$16E_0(E_0-1)$	Z
$*$	1	E_0+1	y_0	$16E_0(E_0-1)$	Z
$*$	1	E_0+1	y_0	$16E_0(E_0-1)$	Z
$*$	1	E_0	y_0	$16(E_0-1)(E_0-2)$	A
$*$	$\frac{1}{2}$	$E_0+\frac{3}{2}$	y_0-1	$4E_0$	λ_T
$*$	$\frac{1}{2}$	$E_0+\frac{3}{2}$	y_0+1	$4E_0$	λ_T
$*$	$\frac{1}{2}$	$E_0+\frac{1}{2}$	y_0-1	$-4E_0+4$	λ_T
$*$	$\frac{1}{2}$	$E_0+\frac{1}{2}$	y_0+1	$-4E_0+4$	λ_T
$*$	0	E_0+1	y_0	$16E_0(E_0-1)$	ϕ

Table 3.7: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ long graviton multiplet with $y_0 \geq 0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name	Mass (^{2})	Name
$*$	$\frac{3}{2}$	E_0+1	y_0	$4E_0-6$	χ^+	$-4E_0-2$	χ^-
$*$	1	$E_0+\frac{3}{2}$	y_0-1	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	Z	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	W
$*$	1	$E_0+\frac{3}{2}$	y_0+1	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	Z	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	W
$*$	1	$E_0+\frac{1}{2}$	y_0-1	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	A	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	Z
$*$	1	$E_0+\frac{1}{2}$	y_0+1	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	A	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	Z
$*$	$\frac{1}{2}$	E_0+2	y_0	$4E_0+2$	λ_T	$-4E_0-2$	λ_L
$*$	$\frac{1}{2}$	E_0+1	y_0-2	$-4E_0+2$	λ_T	$-4E_0-2$	λ_T
$*$	$\frac{1}{2}$	E_0+1	y_0	$-4E_0+2$	λ_T	$4E_0-2$	λ_T
$*$	$\frac{1}{2}$	E_0+1	y_0+2	$-4E_0+2$	λ_T	$4E_0-2$	λ_T
$*$	$\frac{1}{2}$	E_0+1	y_0	$-4E_0+2$	λ_T	$4E_0-2$	λ_T
$*$	$\frac{1}{2}$	E_0	y_0	$4E_0-6$	λ_L	$-4E_0+6$	λ_T
$*$	0	$E_0+\frac{3}{2}$	y_0-1	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	ϕ	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	π
$*$	0	$E_0+\frac{3}{2}$	y_0+1	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	ϕ	$16(E_0-\frac{1}{2})(E_0+\frac{1}{2})$	π
$*$	0	$E_0+\frac{1}{2}$	y_0-1	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	π	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	ϕ
$*$	0	$E_0+\frac{1}{2}$	y_0+1	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	π	$16(E_0-\frac{3}{2})(E_0-\frac{1}{2})$	ϕ

Table 3.8: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ long gravitino multiplets χ^+ and χ^- with $y_0 \geq 0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name	Name	Mass (^{2})	Name
*	1	E_0+1	y_0	$16E_0(E_0-1)$	A	W	$16E_0(E_0-1)$	Z
*	$\frac{1}{2}$	$E_0+\frac{3}{2}$	y_0-1	$-4E_0$	λ_T	λ_L	$4E_0$	λ_T
*	$\frac{1}{2}$	$E_0+\frac{3}{2}$	y_0+1	$-4E_0$	λ_T	λ_L	$4E_0$	λ_T
*	$\frac{1}{2}$	$E_0+\frac{1}{2}$	y_0-1	$4E_0-4$	λ_L	λ_T	$-4E_0+4$	λ_T
*	$\frac{1}{2}$	$E_0+\frac{1}{2}$	y_0+1	$4E_0-4$	λ_L	λ_T	$-4E_0+4$	λ_T
*	0	E_0+2	y_0	$16E_0(E_0+1)$	ϕ	Σ	$16E_0(E_0+1)$	π
*	0	E_0+1	y_0-2	$16E_0(E_0-1)$	π	π	$16E_0(E_0-1)$	ϕ
*	0	E_0+1	y_0+2	$16E_0(E_0-1)$	π	π	$16E_0(E_0-1)$	ϕ
*	0	E_0+1	y_0	$16E_0(E_0-1)$	π	π	$16E_0(E_0-1)$	ϕ
*	0	E_0	y_0	$16(E_0-2)(E_0-1)$	S	ϕ	$16(E_0-2)(E_0-1)$	π

Table 3.9: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ long vector multiplets A,W and Z with $y_0 \geq 0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name
	2	y_0+3	y_0	$16y_0(y_0+3)$	h
	$\frac{3}{2}$	$y_0+\frac{7}{2}$	y_0-1	$-4y_0-12$	χ^-
*	$\frac{3}{2}$	$y_0+\frac{5}{2}$	y_0+1	$4y_0$	χ^+
*	$\frac{3}{2}$	$y_0+\frac{5}{2}$	y_0-1	$4y_0$	χ^+
*	1	y_0+3	y_0-2	$16(y_0+2)(y_0+1)$	Z
	1	y_0+3	y_0	$16(y_0+2)(y_0+1)$	Z
*	1	y_0+2	y_0	$16y_0(y_0+1)$	A
*	$\frac{1}{2}$	$y_0+\frac{5}{2}$	y_0-1	$-4y_0-4$	λ_T

Table 3.10: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ short graviton multiplet with $y_0>0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name
	$\frac{3}{2}$	$y_0+\frac{5}{2}$	y_0	$4y_0$	χ^+
	1	y_0+3	y_0-1	$16(y_0+1)(y_0+2)$	Z
*	1	y_0+2	y_0+1	$16y_0(y_0+1)$	A
*	1	y_0+2	y_0-1	$16y_0(y_0+1)$	A
	$\frac{1}{2}$	$y_0+\frac{5}{2}$	y_0	$-4y_0-4$	λ_T
*	$\frac{1}{2}$	$y_0+\frac{5}{2}$	y_0-2	$-4y_0-4$	λ_T
*	$\frac{1}{2}$	$y_0+\frac{3}{2}$	y_0	$4y_0$	λ_L
*	0	y_0+3	$y_0\pm 1$	$16(y_0+1)(y_0+2)$	ϕ

Table 3.11: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ short gravitino multiplet χ^+ with $y_0>0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name
	1	y_0+2	y_0	$16y_0(y_0+1)$	A
	$\frac{1}{2}$	$y_0+\frac{5}{2}$	$y_0\pm 1$	$-4y_0-4$	λ_T
*	$\frac{1}{2}$	$y_0+\frac{3}{2}$	y_0+1	$4y_0$	λ_L
*	$\frac{1}{2}$	$y_0+\frac{3}{2}$	y_0-1	$4y_0$	λ_L
*	0	y_0+2	y_0-2	$16y_0(y_0+1)$	π
	0	y_0+2	y_0	$16y_0(y_0+1)$	π
*	0	y_0+1	y_0	$16y_0(y_0-1)$	S

Table 3.12: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ short vector multiplet A with $y_0>0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name
*	$\frac{1}{2}$	$y_0+\frac{1}{2}$	y_0-1	$4y_0-4$	λ_L
*	0	y_0+1	y_0-2	$16y_0(y_0-1)$	π
*	0	y_0	y_0	$16(y_0-2)(y_0-1)$	S

Table 3.13: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ hypermultiplet with $y_0>0$

	Spin	Energy	Hypercharge	Mass (^{2})	Name
*	2	3	0	0	h
*	$\frac{3}{2}$	$\frac{5}{2}$	-1	0	χ^+
*	$\frac{3}{2}$	$\frac{5}{2}$	$+1$	0	χ^+
*	1	2	0	0	A

Table 3.14: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ massless graviton multiplet

Using the group–theoretical information of the long graviton multiplet (see table 3.7) we find the energy and hypercharge (E_0, y_0) of the graviton multiplet [12]

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0+36}+\tfrac{1}{2} \\ y_0 &= \frac{2}{3}\left(M_2-M_1\right) , \end{aligned} \tag{3.4.115}$$

and using table 3.7 we find the energies and hypercharges of all the fields in the multiplet. In particular, we see that the gravitinos are in $U(1)_R$ representations $^+$, $^-$, the A, W vectors in $U(1)_R$ representations ^{0}, the Z vectors in $U(1)_R$ representations ^{0}, $^{++}$, $^{--}$. From the mass of the graviton we deduce, using the mass relations (3.1.71), the masses of the gravitinos and vectors present in the graviton multiplet,

$$m_{\chi^\pm} = -6 \pm \sqrt{H_0+36}\,, \tag{3.4.116}$$

$$\begin{aligned} m^2_A &= H_0+48-8\sqrt{H_0+36}\,, \\ m^2_W &= H_0+48+8\sqrt{H_0+36}\,, \\ m^2_Z &= H_0+32\,. \end{aligned} \tag{3.4.117}$$

From equations (3.1.70), we predict the presence of the eigenvalues $M_{(1/2)^3} = m_{\chi^\pm}$ for the spinor. Indeed, looking at (3.4.108), we see that the two eigenvalues λ_1 and λ_2 come from spin–$\frac{3}{2}$ fields that belong to the graviton multiplet. To find out whether there are some

[12] Remember that E_0, y_0 denote the energy and hypercharge of the Clifford vacuum of the multiplet

	Spin	Energy	Hypercharge	Mass (^{2})	Name	Mass (^{2})	Name
*	1	2	0	0	A	0	Z
*	$\frac{1}{2}$	$\frac{3}{2}$	-1	0	λ_L	0	λ_T
*	$\frac{1}{2}$	$\frac{3}{2}$	$+1$	0	λ_L	0	λ_T
*	0	2	0	0	π	0	ϕ
*	0	1	0	0	S	0	π

Table 3.15: M^{111} Kaluza Klein fields in the $\mathcal{N}=2$ massless vector multiplets A and Z

short graviton multiplets present in the spectrum, we now use table 3.4.109. The absence of these eigenvalues λ_1 or λ_2 in some of the exceptional series implies the existence of a short graviton multiplet in that particular G' series. Let us look at it more closely. For instance, for A_2^+ and A_5^+, there is none of the eigenvalues λ_1 or λ_2. This would imply a graviton multiplet without gravitino fields. But fortunately, the series A_2 and A_5 do not contain representations of G' in which there is a graviton field, see (3.4.114). Considering the rest of table 3.4.109 and also table 3.4.112, we find three types of graviton multiplets: a long graviton multiplet and two types of short graviton multiplets. The long graviton multiplet contains four spinors χ: χ^+ with hypercharge $y_0 \pm 1$ and χ^- with hypercharge $y_0 \pm 1$. They are found in the G' representations of $A_R, A_1, A_1^*, A_3, A_3^*, A_6, A_7$. Then there is a short graviton multiplet in the series A_4 and A_4^*. From tables 3.4.109 and 3.4.112, one sees that they contain the two χ^+ with hypercharge $y_0 \pm 1$, but only one χ^-, i.e. for A_4 we have one χ^- with $y_0 - 1$, and for A_4^* we have one χ^- with $y_0 + 1$. We also find the massless multiplet in A_8 for which none of the spin–$\frac{3}{2}$ fields χ^- are present.

At this stage, we know that the spin–$\frac{3}{2}$ fields that correspond to the eigenvalues λ_1 and λ_2 in (3.4.108) and (3.4.111) sit in the graviton multiplets. However, there are also spin–$\frac{3}{2}$ fields that yield the eigenvalues λ_3 and λ_4 in (3.4.108) and (3.4.111). They can only be gravitinos of the gravitino multiplets in the spectrum. So now we know the highest components of gravitino multiplets, their energies, hypercharges and G' representations. But before we continue with the gravitino multiplet, let us look at the vectors of the graviton multiplet.

Let us consider A and W first. We know that, if present, they should be in the series (3.4.114). Using equations (3.1.70) we see that their $M_{(1)(0)^2}$ eigenvalues would then be

$$\begin{aligned} M^A_{(1)(0)^2} &= H_0 + 24 + 4\sqrt{H_0+36}\,, \\ M^W_{(1)(0)^2} &= H_0 + 24 - 4\sqrt{H_0+36}\,. \end{aligned} \tag{3.4.118}$$

Indeed, these eigenvalues are present, namely for A we find λ_4 and for W we find λ_5 of eq. (3.4.83). To determine whether, in the exceptional series, the vector A or the vector W is present we use table 3.4.86. The absence of one of the vectors will imply shortening of the graviton multiplet. Studying the spin 3/2 fields, we have found that there are long graviton multiplets in the series $A_R, A_1, A_1^*, A_3, A_3^*, A_6, A_7$ and short graviton multiplets in the series A_4, A_4^* . This is confirmed here: in the former series both the A and W fields are present, in the latter only the field A is present. For the massless multiplet of A_8 we also see that only the vector A is present.

Let us look at the vector Z in the graviton multiplet. We know that the Z vectors should be in the same G' representations of the graviton:

$$A_R, A_1, A_1^*, A_3, A_3^*, A_4, A_4^*, A_6, A_7, A_8 \tag{3.4.119}$$

and that two Z vectors should be in the series ^{0}, one in the series $^{++}$ and one in the series $^{--}$. For the operator $M_{(1)^2(0)}$ on the two–form we predict, using eq.s (3.1.70), the presence of the eigenvalue

$$M^Z_{(1)^2(0)} = H_0 + 32\,. \tag{3.4.120}$$

Indeed, it corresponds to λ_{10} and λ_{11} in (3.4.91) for the series ^{0}, and λ_4 in (3.4.93) for the series $^{++}$ (and $^{--}$, which are the series of the conjugate representations of $^{++}$ $(M_2 \leftrightarrow M_1)$).

So we see that for the long graviton multiplets all the vectors Z are present. Using the fact that

$$\begin{aligned} B_R \cup B_4 \cup B_5 \cup B_6 \cup B_7 \cup B_8 \cup B_9 \cup B_{10} &= A_R \cup A_1 \cup A_1^* \cup A_3 \cup A_3^* \cup A_4 \cup A_6 \cup A_7, \\ B_R^* \cup B_4^* \cup B_5^* \cup B_6^* \cup B_7^* \cup B_8^* \cup B_9^* \cup B_{10}^* &= A_R \cup A_1 \cup A_1^* \cup A_3 \cup A_3^* \cup A_4^* \cup A_6 \cup A_7, \end{aligned} \tag{3.4.121}$$

and tables 3.4.92 and 3.4.94 we find that for the short graviton multiplets of A_4 we have two Z's, one with hypercharge y and one with hypercharge $y-2$; for the short graviton multiplets of A_4^* we have two Z's, one with hypercharge y and one with hypercharge $y+2$; for the massless graviton multiplet we have no vectors Z.

To determine which λ_T fields and scalar fields ϕ are present, we use the $\mathcal{N}=2 \to \mathcal{N}=1$ decomposition of the multiplets (2.4.66),. . .,(2.4.78). We already know where λ_T and ϕ are located in the long graviton multiplet from table 3.7 [23]. From the decomposition of the long $\mathcal{N}=2$ graviton multiplet (2.4.66) we see that it is made of four $\mathcal{N}=1$ massive multiplets: one graviton, two gravitino and a vector multiplet. Harmonic analysis teaches us that in the short graviton multiplet there are three gravitino fields and three vector fields. The only possible structure of the short graviton multiplet is then the one displayed in chapter 2 and in table 3.10.

The multiplet that we have found in the representation of series A_8 is in fact the massless graviton multiplet. In this case the field A becomes the graviphoton. The final structure of the short graviton multiplet and the massless graviton multiplet is displayed in tables 3.10 and 3.14 respectively.

The gravitino multiplet

As already previously explained, we know the $M_{(1/2)^3}$ eigenvalues and the G representations of the spin–$\frac{3}{2}$ in the gravitino multiplet from the matching of the graviton multiplet. Their masses are given by equations (3.1.70),

$$\begin{aligned} m_{\chi^+} &= -8 + \sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} = \lambda_3 \\ m_{\chi^-} &= -8 - \sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} = \lambda_4 \end{aligned} \tag{3.4.122}$$

for series of type $^+$ and

$$\begin{aligned} m_{\chi^+} &= -8 + \sqrt{H_0 + 16 - \tfrac{32}{3}(M_2 - M_1)} = \lambda_3 \\ m_{\chi^-} &= -8 - \sqrt{H_0 + 16 - \tfrac{32}{3}(M_2 - M_1)} = \lambda_4 \end{aligned} \tag{3.4.123}$$

for series of type $^-$. Each of the above four different eigenvalues gives rise to gravitino multiplets of different types and/or in different G' representations. Now we look at tables 3.4.109 and 3.4.112 and see that we have gravitino multiplets for the series $A_R^{\pm}$ and $A_1^{\pm}$. We consider the gravitino multiplets in the series of type $^+$ only. The gravitino multiplets in the series of type $^-$ coming from (3.4.123) can be obtained be taking the conjugates of the gravitino multiplets in the series of type $^+$.

We start with χ^+ in the series of type $^+$. The energy and hypercharge $(E_0,\ y_0)$ of the gravitino multiplets are given by,

$$
\begin{aligned}
E_0 &= \tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} - \tfrac{1}{2} \\
y_0 &= \frac{2}{3}\,(M_2 - M_1) - 1\,.
\end{aligned}
\tag{3.4.124}
$$

Let us look at the vectors in the gravitino multiplets. As we know from group theory (see table 3.8) we should find a vector with hypercharge y_0+1 and energy $E_0+\frac{1}{2}$, in the series ^{0}. However group theory does not tell us whether it is the vector A or the vector W. But since we know that in series of type $^+$ we have $m_{\chi^+} \geq -8$, we can use the mass relations (3.1.71) to derive

$$
m_A^2 = H_0 + \tfrac{32}{3}(M_2 - M_1) + 48 - 12\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16} \tag{3.4.125}
$$

or

$$
m_W^2 = m_\chi^2 + 2m_\chi + 192\,. \tag{3.4.126}
$$

We see from table 3.8 that it is the A vector which is present in the χ^+ gravitino multiplet and not W. Hence, comparing with the formula (3.1.70) in order to find A, we expect the following eigenvalue

$$
M^A_{(1)(0)^2} = H_0 + \tfrac{32}{3}(M_2 - M_1) \tag{3.4.127}
$$

for the $M_{(1)(0)^2}$ operator. Looking at table 3.4.83we see that it is indeed present: λ_1. Looking at table 3.4.86 we see that it appears in the series $A^0_R, A^0_1, A^0_2, A^0_3, A^0_4, A^0_5$. We also find a vector A with hypercharge $y_0 - 1$ in series $^{++}$. Indeed, using

$$
B_R \cup B_1 \cup B_3 \cup B_4 \cup B_6 \cup B_7 = A_R \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5\,, \tag{3.4.128}
$$

we see that (3.4.127) is an eigenvalue of the one–form operator $M_{(1)(0)^2}$ in series $^{++}$ as given in (3.4.87). Both the spin–1 fields A with $y_0 - 1$ and $y_0 + 1$ of the gravitino multiplet for χ^+ are present and there are no other left with eigenvalue (3.4.127). For the vector Z sector, we expect the presence of two states with mass

$$
m_Z^2 = H_0 + 16 + \tfrac{32}{3}(M_2 - M_1) - 4\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)}, \tag{3.4.129}
$$

one in the G representations of type ^{0}, the other in the representations $^{++}$ or $^{--}$ (depending on the G representation of the gravitino). The mass (3.4.129) corresponds to λ_7 in (3.4.91) and λ_3 in (3.4.93). From this we see that Z is present except for series A_5, and series B_7. The series A_5 and B_7 have no overlap. So we conclude that we have long gravitino multiplets except if the multiplet sits in a representation of A_5 or B_7. For the gravitino multiplet with χ^+ in the series $^+$, we now look at the mass of the scalar π,

$$
m_\pi^2 = 16\,(\tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}} - 1)(\tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}} - 2)\,. \tag{3.4.130}
$$

From eq.s (3.1.70) we predict the eigenvalue

$$
M^\pi_{(1)^3} = \tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} \tag{3.4.131}
$$

which we do find as λ_1 in (3.4.99) in series $A^0_R, A^0_1, A^0_2, A^0_3, A^0_4$ (see (3.4.100)) and as λ_1 in (3.4.103) in the series $B^{++}_R, B^{++}_1, B^{++}_3, B^{++}_4, B^{++}_6$ (see (3.4.104)). So none of the fields π with y_0-1 and y_0+1 is present in the short gravitino multiplets with χ^+ in the series of type $^+$. Let us now consider the spin$-\frac{1}{2}$ field λ^+_L. Looking at the expansion (3.1.61), we see that λ_L appears in the expansion of the spinor. So we can check whether it is present in the gravitino multiplet with χ^+ in the series $^+$. Its mass is (3.1.70)

$$m_{\lambda^+_L} = -8 + \sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)}\,, \tag{3.4.132}$$

so, from eq.s (3.1.70) we expect the eigenvalue

$$M^{\lambda^+_L}_{(1/2)^3} = -8 - \sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} \tag{3.4.133}$$

which we do find as λ_4 in (3.4.108) in $A^+_R, A^+_1, A^+_2, A^+_3, A^+_4, A^+_5$ (see (3.4.109)). So the field λ^+_L is present in both long and short gravitino multiplets with hypercharge y_0. In fact it has to be there since it provides the Clifford vacuum of the representation. For the short gravitino multiplets we have found which of the fields ϕ and λ_T are present by using the $\mathcal{N}=2 \to \mathcal{N}=1$ decomposition (2.4.73) and by calculating the norms of the states (see chapter 2). The result is displayed in table 3.11.

Let us consider χ^- for the series of type $^+$. It has mass m_{χ^-} from (3.4.122). The energy and hypercharge $(E_0,\ y_0)$ of the multiplet are

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} + \tfrac{3}{2}\,. \\ y_0 &= \frac{2}{3}\,(M_2 - M_1) - 1\,. \end{aligned} \tag{3.4.134}$$

We now have $m_\chi \leq -8$. So, using the mass relations for W we find

$$m^2_W = H_0 + \tfrac{32}{3}(M_2 - M_1) + 48 + 12\sqrt{H_0 + \tfrac{32}{3}(M_2 - M_1) + 16}\,. \tag{3.4.135}$$

Thus in this case it is W that is present and not A. We find the same eigenvalue (3.4.127), so we conclude that W is present in all types of gravitino multiplets with χ^- in series of type $^+$. For Z we have

$$m^2_Z = H_0 + 16 + \tfrac{32}{3}(M_2 - M_1) + 4\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} \tag{3.4.136}$$

which, according to eq.s (3.1.70), has to be an eigenvalue of the two–form mass operator. Indeed, for series of type ^{0} it corresponds to λ_6, which is present in series $A_R, A_1, A_2, A_3, A_4, A_5$ (see (3.4.92)). Notice that these are the same series of representations as the ones in which we found χ^+. For the series $^{++}$ we find λ_2, which is present in the series $B_R, B_1, B_3, B_4, B_6, B_7$ (see (3.4.94)), which are again the same series of representations as for χ^+. The fields π present have mass,

$$m^2_\pi = 16\,(-\tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} - 1)(-\tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} - 2)\,. \tag{3.4.137}$$

So we predict the eigenvalue

$$M^{\pi}_{(1)^3} = -\tfrac{1}{4}\sqrt{H_0 + 16 + \tfrac{32}{3}(M_2 - M_1)} \tag{3.4.138}$$

Indeed it is λ_3 in (3.4.99), present in the series $A_R, A_1, A_2, A_3, A_4, A_5$ (3.4.100) and λ_2 in (3.4.104), present in the series $B_R, B_1, B_3, B_4,\ B_6, B_7$ (3.4.103). We conclude that all the gravitino multiplets with χ^- are long gravitino multiplets.

The vector multiplet

What are the vector field we have been left with? They have to be the highest components of the vector multiplets. Well, we have a multiplet with highest component vector A with eigenvalue λ_5 in (3.4.83). We have a vector multiplet with highest vector component W with eigenvalue λ_4 in (3.4.83). We have some vector multiplets with highest vector component Z with eigenvalues λ_3 in (3.4.91), λ_5 in (3.4.93) and λ_5^* in the series $^{--}$. All these eigenvalues give rise to the existence of different types of vector multiplets in different representations of G'.

Let us start with A. We call this the A–vector multiplet. It has eigenvalue λ_5 in (3.4.83). Its energy and hypercharge are

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0+36}-\tfrac{3}{2} \\ y_0 &= \frac{2}{3}\left(M_2-M_1\right) \end{aligned} \tag{3.4.139}$$

and the mass of the field component A is

$$m_A^2 = H_0 + 96 - 16\,\sqrt{H_0+36}\,. \tag{3.4.140}$$

This eigenvalue is present in the series $A_R^0, A_1^0, A_1^{*0}, A_3^0, A_3^{*0}, A_6^0, A_7^0$. We now figure out for which of these there is shortening. From the table 3.9 we see that π has the same mass as A (3.4.140), and using eq.s (3.1.70) we conclude that we should find the eigenvalues

$$M^{\pi}_{(1)^3} = \tfrac{1}{4}\sqrt{H_0+36}-\tfrac{1}{2}\,, \tag{3.4.141}$$

which is present: λ_5 in $A_R^0, A_1^0, A_1^{0*}, A_3^0, A_3^{0*}, A_6^0, A_7^0$ (3.4.99) (3.4.100). It is also present as λ_3 in $B_R^{++}, B_4^{++}, B_5^{++}, B_6^{++}, B_7^{++}, B_9^{++}$ (3.4.103) (3.4.104). Considering (3.4.121) this seems strange at first sight. However, what happens is that here we discover a scalar π in the series A_4 of a hypermultiplet. We can see this as follows. Suppose the eigenvalue were also present in series B_8 and series B_{10}. Then the eigenvalue λ_3 would appear in the representations of B that are on the right–hand side of (3.4.121). So we would find the field π in the G' representations $A_R, A_1, A_1^*, A_3, A_3^*, A_6, A_7$ and in A_4, with $Y = \frac{2}{3}(M_2-M_1)-2$. The series A_4 and B_8 and B_{10} have no overlap. Consequently, the π in A_4 can not belong to the A–vector multiplet and thus has to be a scalar of a hypermultiplet. Similarly, we find π in $B_R^{--*}, B_4^{--*}B_5^{--*}, B_6^{--*}, B_7^{--*}, B_8^{--*}, B_9^{--*}, B_{10}^{--*}$. With the same reasoning, we conclude that π in A_4^* with $Y = \frac{2}{3}(M_2-M_1)+2$ has to be a scalar of some hypermultiplet. However, λ_3 does not sit in the series $B_8, B_8^*, B_{10}, B_{10}^*$. So we conclude that we get shortening in these series. Now we get different types of short vector multiplets. This is due to fact the B_8 and B_8^* have overlap, namely if $M_1 = M_2 = 1, J = 0$ and that also B_{10} and B_{10}^* have overlap, namely for the representation $M_1 = M_2 = 0, J = 1$. For the representations in the series B_8 and B_{10} with $M_1 > M_2 = 1$, we find that the field π with hypercharge $y-2$ in the long vector multiplet decouples. The representations

$$\begin{aligned} M_1 = M_2, &\quad J = 1 \\ M_1 = M_2 = 1, &\quad J = 0 \end{aligned} \tag{3.4.142}$$

yield massless vector multiplets. They contain the vectors that gauge $SU(2)$ and $SU(3)$ respectively.

Let us now figure out whether we can learn something about the presence of ϕ, S and Σ in the A–vector multiplet. The table 3.13 gives the mass,

$$m^2_{\phi,S/\Sigma} = 16\,E_0(E_0+1) = H_0 + 48 - 4\sqrt{H_0+36}\,. \tag{3.4.143}$$

Looking at eq.s (3.1.70), we see that the entry in the table can not be S or Σ, but has to be ϕ. If we look at the other $\phi, S/\Sigma$ in the table with mass

$$m^2_{\phi,S/\Sigma} = 16\,(E_0-2)(E_0-1) = H_0 + 176 - 24\sqrt{H_0+36}\,, \tag{3.4.144}$$

we see that it is the mass for the field S. So at this place in the table we find the field S. The field S is found in the series $A^0_R, A^0_1, A^{0*}_1, A^0_3, A^{0*}_3, A^0_4, A^{0*}_4, A^0_6, A^0_7, A^0_8$. So it is always present in the A–vector multiplets. Besides, we get some extra S–fields that are to be put in the hypermultiplets in the series A_4, A^*_4, A_8.

To conclude the discussion of the A vector multiplet, there is shortening of A–vector multiplets in series B_8, B^*_8 and B_{10}, B^*_{10}. In the representation (3.4.142) there are massless vector multiplets, in the other $B_8, B^*_8, B_{10}, B^*_{10}$ representations there are short vector multiplets. The ϕ and λ_T contents of the short vector multiplets can be determined by using the $\mathcal{N}=2 \to \mathcal{N}=1$ decomposition (2.4.74). The structure of the long vector multiplet and the short vector multiplet is displayed in table 3.9 and 3.12 respectively.

Let us now consider the vector multiplet with highest vector component W. We will call this the $W-$ vector multiplet. We expect eigenvalue λ_5 in (3.4.83) and (3.4.86), which we find in series $A^0_R, A^0_1, A^{0*}_1, A^0_3, A^{0*}_3, A^0_4, A^{0*}_4, A^0_6, A^0_7, A^0_8$. This multiplet has energy and hypercharge,

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0+36} + \tfrac{5}{2}\,, \\ y_0 &= \frac{2}{3}\,(M_2 - M_1)\,, \end{aligned} \tag{3.4.145}$$

the W field has mass

$$m^2_W = H_0 + 96 + 16\sqrt{H_0+36}\,. \tag{3.4.146}$$

For the fields π, we expect to find the eigenvalues λ_6 in series A^0_R, A^0_1, A^{*0}_1, A^0_3, A^{*0}_3, A^0_4, A^{*0}_4, A^0_6, A^0_7, A^0_8 (3.4.99), (3.4.100), and λ_4 in series B^{++}_R, B^{++}_4, B^{++}_5, B^{++}_6, B^{++}_7, B^{++}_8, B^{++}_9, B^{++}_{10}, B^{++}_{11} (3.4.103), (3.4.104), and λ^*_4 in series B^{--}_R, B^{--}_4, B^{--}_5, B^{--}_6, B^{--}_7, B^{--}_8, B^{--}_9, B^{--}_{10}, B^{--}_{11}. Using

$$\begin{aligned} B_{11} &= A_4 \cup A_8\,, \\ B^*_{11} &= A^*_4 \cup A_8\,, \end{aligned} \tag{3.4.147}$$

and (3.4.121), we see that all these ^{0}, $^{++}$, and $^{--}$ series coincide. Thus all the fields π in the table of [23] are always present and we find no fields π that have to be put in other multiplets. So the W–vector multiplet is always long. Which of the fields $\phi, S/\Sigma$ are present? Let us look at $\phi, S/\Sigma$ with mass

$$m^2_{\phi,S/\Sigma} = 16\,E_0(E_0+1) = H_0 + 176 + 24\sqrt{H_0+36}\,. \tag{3.4.148}$$

From eq.s (3.1.70) we see that it is the field Σ that is present in the series A^0_R, A^0_1, A^{0*}_1, A^0_3, A^{0*}_3, A^0_4, A^{0*}_4, A^0_6, A^0_7, A^0_8. So this confirms that there is no shortening and we do not

find any extra fields Σ that are to be put in the hypermultiplets. Let us look at $\phi, S/\Sigma$ with mass

$$m^2_{\phi,S/\Sigma} = 16\,(E_0-2)(E_0-1) = H_0 + 48 + 8\sqrt{H_0+36}\,. \tag{3.4.149}$$

This can only be the field ϕ. So we conclude that the W–vector multiplets are always long vector multiplets. And there are no scalar left that have to be put in hypermultiplets. Its structure is displayed in table 3.9.

Let us now look at the Z–vector multiplet with eigenvalue λ_3 in series A_R, A_1, A_1^*, A_6, A_8 (3.4.91) (3.4.92). The multiplet has energy and hypercharge

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0+4} + \tfrac{1}{2}\,, \\ y_0 &= \frac{2}{3}\,(M_2 - M_1)\,, \end{aligned} \tag{3.4.150}$$

the field Z has mass

$$m^2_Z = H_0\,. \tag{3.4.151}$$

What about the two fields π? Let us look at π with mass

$$m^2_\pi = 16\,E_0(E_0+1) = H_0 + 16 + \sqrt{H_0+4}\,. \tag{3.4.152}$$

From eq.s (3.1.70) we expect there to be λ_7 in (3.4.99). Indeed, it is present in series $A^0_R, A^0_1, A^{*0}_1, A^0_6$. So we get shortening in the singlet representation A_8. For π with mass

$$m^2_\pi = 16\,(E_0-2)(E_0-1)\,, \tag{3.4.153}$$

we find λ_8 in series $A_R,^0 A^0_1, A^{0*}_1, A^0_6, A^0_8$. So finally, we conclude that for this type of Z–vector multiplet (with λ_3 in (3.4.91)) there is shortening in series A_8, which yields the massless Betti multiplet. The structure of the long Z–vector multiplet and the massless Betti multiplet is displayed in tables 3.9 and 3.15 respectively.

Let us now look at the Z–vector multiplet with λ_5 in (3.4.93). It appears in series B_R, B_1, B_2 (3.4.94). The multiplet has energy and hypercharge

$$\begin{aligned} E_0 &= \tfrac{1}{4}\sqrt{H_0 + \tfrac{64}{3}(M_2-M_1) - 28} + \tfrac{1}{2} \\ y_0 &= \frac{2}{3}\,(M_2 - M_1) - 2, \end{aligned} \tag{3.4.154}$$

the field Z has mass

$$m^2_Z = H_0 + \tfrac{64}{3}(M_2-M_1) - 32\,. \tag{3.4.155}$$

What about the presence of the fields π? For π with mass

$$m^2_\pi = 16\,(E_0-2)(E_0-1)\,, \tag{3.4.156}$$

we expect the eigenvalue λ_5 in (3.4.103), which is found in the series $B^{++}_R, B^{++}_1, B^{++}_2$ (3.4.104). For π with mass

$$m^2_\pi = 16\,E_0(E_0+1)\,, \tag{3.4.157}$$

we expect λ_6 in (3.4.103), which is found in the series $B^{++}_R, B^{++}_1, B^{++}_2$ (3.4.104). So we conclude that for the Z–vector multiplet (with vector Z with eigenvalue λ_5 in (3.4.93)), there is never shortening. We do not find extra scalars that are to be put in hypermultiplets either. The structure of this long Z vector multiplet is displayed in table 3.9.

For the Z–vector multiplet with λ^*_5 in series B^*_R, B^*_1, B^*_2, one just takes the conjugate of the previous results.

The hypermultiplet

After having put the scalars π in the right places in the graviton, the gravitino and the vector multiplet, we are only left with scalars π in series A_4^0 and A_4^{0*} and S in series $A_4,^0$ A_4^{0*}, A_8^0.

So for each representation of A_4 we find a hypermultiplet with energy

$$E_0 = \tfrac{1}{4}\sqrt{H_0+36} - \tfrac{3}{2} \tag{3.4.158}$$

containing the field π with hypercharge $Y = \frac{2}{3}(M_2 - M_1) - 2$ and mass

$$m_\pi^2 = H_0 + 96 - 16\,\sqrt{H_0+36} \tag{3.4.159}$$

and the field S with $Y = \frac{2}{3}(M_2 - M_1)$ and mass

$$m_S^2 = H_0 + 176 - 24\,\sqrt{H_0+36}\,. \tag{3.4.160}$$

The scalars of this hypermultiplet are complete if we add the scalars π and S of A_4^*, which are in fact the complex conjugates of the scalars in A_4. From the eigenvalues of the operator $M_{(1/2)^3}$ we find the λ_L necessary to fill all the hypermultiplets. The structure of the hypermultiplets is displayed in the table 3.13.

In order to correctly match the fields with the multiplets, it is important to note that in the singlet G representation $M_1 = M_2 = J = Y = 0$ the scalar S is absent. This is due to the fact that, from the Kaluza Klein expansion (3.1.61) of the eleven-dimensional field $h_{mn}(x,y)$, the scalar S appears in the expressions $(6 - \sqrt{M_{(0)^3}+36})S^I(x)$ and $\mathcal{D}_{(m}\mathcal{D}_{n)}(2+\sqrt{M_{(0)^3}+36})S^I(x)$. The coefficient of the former, $6-\sqrt{M_{(0)^3}+36}$, disappears in the singlet representation. The latter become a pure gauge term, due to the freedom of coordinate reparametrization, being the graviton in the singlet G representation the massless graviton.

At this point we have done the complete matching of the multiplets with the spectrum of Laplace Beltrami operators. It is reassuring that all the fields we have found have been organized in $\mathcal{N}=2$ AdS_4 multiplets. An important result is that we have established the existence of short multiplets. From the expressions of the energies and hypercharges (E_0, y_0) we have found, we can easily derive that what we expect on unitarity bounds and shortening conditions is confirmed:

- for all the long multiplets

$$E_0 > |y_0| + s_0 + 1$$

- for all the short graviton, gravitino and vector multiplets

$$E_0 = |y_0| + s_0 + 1$$

- for all the hypermultiplets

$$E_0 = |y_0| \geq \frac{1}{2}$$

- for all the massless multiplets

$$E_0 = s_0 + 1 \quad y_0 = 0.$$

3.5 The mass spectra of $AdS_4 \times N^{010}$ and $AdS_4 \times Q^{111}$ supergravities

3.5.1 N^{010}

Here I do not review the harmonic analysis on N^{010}, worked out in [53] (see also, for the geometry, [59]). I simply give the result, namely, the spectrum of $Osp\,(3|4)$ multiplets [17]. In this case we have

$$G = SU\,(3) \times SU\,(2) = G' \times SU\,(2) \tag{3.5.1}$$

where $SU\,(2)$ is the R–symmetry. The $\mathcal{N} = 3$ supermultiplets are then organized in $SU\,(3)$ UIRs, which I denote as usual with the Young labels M_1, M_2, while J denotes the isospin (see chapter 2) of a field in a supermultiplet.

Long multiplets

There are long multiplets for the following $SU\,(3)$ representations:

$$\begin{cases} M_1 = k & k \geq 0 \\ M_2 = k + 3j & j \geq 0 \end{cases} \tag{3.5.2}$$

k, j integers.

- For every $SU\,(3)$ representation with $k \geq 0,\ j \geq 2$ there is only one of the following multiplets, that are long:

multiplet	*isospin*	energy
$SD\,(E_0, 2, J_0)$	$j \leq J_0 \leq k + j$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36}$
$SD\,(E_0, 3/2, J_0)$	$j \leq J_0 \leq k + j$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} - \frac{3}{2}$
$SD\,(E_0, 3/2, J_0)$	$j \leq J_0 \leq k + j$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} + \frac{3}{2}$

(3.5.3)

- For every $SU\,(3)$ representation with $k \geq 0,\ j = 1$ there is only one of the following multiplets, that are long:

multiplet	*isospin*	energy
$SD\,(E_0, 2, J_0)$	$1 \leq J_0 \leq k + 1$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36}$
$SD\,(E_0, 3/2, J_0)$	$1 \leq J_0 < k + 1$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} - \frac{3}{2}$
$SD\,(E_0, 3/2, J_0)$	$1 \leq J_0 \leq k + 1$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} + \frac{3}{2}$

(3.5.4)

- For every $SU\,(3)$ representation with $k \geq 0,\ j = 0$ there is only one of the following multiplets, that are long:

multiplet	*isospin*	energy
$SD\,(E_0, 2, J_0)$	$0 \leq J_0 < k$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36}$
$SD\,(E_0, 3/2, J_0)$	$0 \leq J_0 < k$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} - \frac{3}{2}$
$SD\,(E_0, 3/2, J_0)$	$0 \leq J_0 \leq k$	$E_0 = \frac{1}{4}\sqrt{H_0 + 36} + \frac{3}{2}$

(3.5.5)

Short multiplets

There are the following short multiplets in the following $SU(3)$ representations:

- There is only one massive short graviton multiplet $SD(J_0 + 3/2, 2, J_0)$ in each of the representations:
$$M_1 = k, \quad M_2 = k, \quad k \geq 1. \tag{3.5.6}$$
It has
$$E_0 = k + 3/2, \; J_0 = k. \tag{3.5.7}$$

- There is only one massive short gravitino multiplet $SD(J_0 + 1, 3/2, J_0)$ in each of the representations:
$$M_1 = k, \quad M_2 = k + 3, \quad k \geq 0. \tag{3.5.8}$$
It has
$$E_0 = k + 2, \; J_0 = k + 1. \tag{3.5.9}$$

- There is only one massive short vector multiplet $SD(J_0, 1, J_0)$ in each of the representations:
$$M_1 = k, \quad M_2 = k, \quad k \geq 2. \tag{3.5.10}$$
It has
$$E_0 = k, \; J_0 = k. \tag{3.5.11}$$

Massless multiplets

The massless sector of the theory is composed by the following multiplets.

- There is one massless graviton multiplet in the representation:
$$M_1 = M_2 = 0\,. \tag{3.5.12}$$
It has
$$E_0 = 3/2, \; J_0 = 0. \tag{3.5.13}$$
This multiplet has the standard field content expected for the $\mathcal{N} = 3$ supergravity multiplet in four–dimensions, namely one massless graviton, three massless gravitinos that gauge $\mathcal{N} = 3$ supersymmetry, three massless vector fields (organized in a $J_0 = 1$ adjoint representation of $SO(3)_R$) that gauge the R-symmetry and one spin one–half field.

- There is one massless vector multiplet in each of the representations:
$$M_1 = M_2 = 1 \tag{3.5.14}$$
$$M_1 = M_2 = 0\,. \tag{3.5.15}$$
They have:
$$E_0 = 1, \; J_0 = 1. \tag{3.5.16}$$
The multiplet (3.5.14) contains the gauge vectors of the $SU(3)$ isometry. The multiplet (3.5.15) is the Betti multiplet [8], related to the non–trivial cohomology of N^{010} in degree two.

It is worth noting that before the harmonic analysis on N^{010} was performed, the complete structure of short $\mathcal{N}=3$ supermultiplets was not known (only the short vector multiplet structure had been derived [24]). As in the M^{111} case, the partial knowledge of $Osp\,(3|4)$ UIRs joined with harmonic analysis of part of the operators (3.1.63),. . .,(3.1.69) yielded both the complete spectrum of $AdS_4 \times N^{010}$ supergravity given above and the complete structure of $\mathcal{N}=3$ UIRs given in chapter 2.

3.5.2 Q^{111}

The harmonic analysis of

$$\begin{aligned} Q^{111} &= \frac{SU\,(2) \times SU\,(2) \times SU\,(2)}{U\,(1) \times U\,(1)} \\ &= \frac{SU\,(2) \times SU\,(2) \times SU\,(2) \times U\,(1)}{U\,(1) \times U\,(1) \times U\,(1)} \end{aligned} \tag{3.5.17}$$

has not been carried out yet (at the moment it is work in progress [18]). So the complete spectrum of supergravity on $AdS_4 \times Q^{111}$ is not known at the moment. Nevertheless the spectrum of the scalar operator (3.1.63), namely the laplacian $\mathcal{D}^a\mathcal{D}_a$, has been found long ago by C. Pope [44], not by harmonic analysis but by means of explicit resolution of the differential equations.

First of all an explicit coordinatization of the manifold Q^{111} has been found, by noting that Q^{111} is an $U\,(1)$ fiber bundle on $S^2 \times S^2 \times S^2$. In the next chapter I will go into detail of this coordinate description, for Q^{111} and M^{111}.

Then, in terms of this coordinate system, the eigenvalues of

$$\mathcal{D}^a\mathcal{D}_a\mathcal{H}\,(y) = M_{(0)^3}\mathcal{H}\,(y) \tag{3.5.18}$$

have been found. Here I only give the result of this calculation, which will be an useful hint for the construction of the dual conformal theory in the next chapter.

I remind that the labels of the $G' = (SU\,(2))^3$ UIRs are given by the three $SU\,(2)$-spins

$$\left[J^{(1)}, J^{(2)}, J^{(3)}\right]. \tag{3.5.19}$$

For each value of the three labels (3.5.19), integer of half integer (differently from the case of M^{111}, where only integer $SU\,(2)$ spins were allowed), there is an eigenvalue of the scalar operator, which is

$$M_{(0)^3} = 32\left(J^{(1)}\left(J^{(1)}+1\right) + J^{(2)}\left(J^{(2)}+1\right) + J^{(3)}\left(J^{(3)}+1\right)\right). \tag{3.5.20}$$

Then, looking at the mass formula (3.1.70), we find that for every G' representation $\left[J^{(1)}, J^{(2)}, J^{(3)}\right]$ there are the following AdS_4 fields:

- one graviton field $h_{mn}\,(x)$ with mass squared $m_h^2 = M_{(0)^3}$;
- one scalar field $S\,(x)$ with mass squared $m_\Sigma^2 = M_{(0)^3} + 176 + 24\sqrt{M_{(0)^3}+36}$;
- one scalar field $\Sigma\,(x)$ with mass squared $m_S^2 = M_{(0)^3} + 176 - 24\sqrt{M_{(0)^3}+36}$;

but we do not know anything about the fields $\phi(x)$, $\pi(x)$, $W(x)$, $A(x)$, $Z(x)$, $\lambda_L(x)$, $\lambda_T(x)$, $\chi(x)$.

Notice, however, that from the table 3.13, found by studying the M^{111} spectrum but having a more general validity, we see that every hypermultiplet (and then chiral superfield) has a field S as lowest energy field; it is then reasonable that the chiral superfields (which, as we will see in next chapter, are the fundamental degrees of freedom of the conformal theory on the boundary) are in the flavour representations

$$J^{(1)} = J^{(2)} = J^{(3)} = k/2 \quad k \in \mathbb{Z}. \tag{3.5.21}$$

Chapter 4

Superconformal field theories dual to $AdS_4 \times \left(\frac{G}{H}\right)_7$ supergravities

The purpose of this chapter is to determine the conformal theory on a collection of M2-branes sitting at the singular point of the cone (1.2.12) $\mathcal{C}(X_7)$ (here named *conifold*), where $X_7 = Q^{111}$ or $X_7 = M^{111}$ [1]. Such a theory is dual, by AdS/CFT correspondence, to the supergravities on $AdS_4 \times Q^{111}$ and $AdS_4 \times M^{111}$, which have been studied in chapter 3. If we find such a theory, this would be a strong check to AdS/CFT correspondence.

While for branes sitting at orbifold singularities there is a straightforward method for identifying the conformal theory living on the world-volume [62], [63], for conifold singularities much less is known [64], [65]. The strategy of describing the conifold as a deformation of an orbifold singularity used in [13], [65] and identifying the superconformal theory as the IR limit of the deformed orbifold theory, seems more difficult to be applied in three dimensions [2]. We will then use the intuition from geometry in order to identify the fundamental degrees of freedom of the superconformal theory and to compare them with the results of the KK expansion.

We expect to find the superconformal fixed points dual to AdS-compactifications as the IR limits of three-dimensional gauge theories. In the maximally supersymmetric case $AdS_4 \times S^7$, for example, the superconformal theory is the IR limit of the $\mathcal{N} = 8$ supersymmetric gauge theory [1]. In three dimensions, the gauge coupling constant is dimensionful and a gauge theory is certainly not conformal. However, the theory becomes conformal in the IR, where the coupling constant blows up. In this simple case, the identification of the superconformal theory living on the world-volume of the M2-branes follows from considering M-theory on a circle. The M2-branes become D2-branes in type IIA, whose world-volume supports the $\mathcal{N} = 8$ gauge theory with a dimensionful coupling constant related to the radius of the circle. The near horizon geometry of D2-branes is not anymore AdS [63], since the theory is not conformal. The AdS background and conformal invariance is recovered by sending the radius to infinity; this corresponds to sending the gauge theory coupling to infinity and probing the IR of the gauge theory.

We expect a similar behaviour for other three dimensional gauge theories. As a difference with four–dimensional CFT's corresponding to AdS_5 backgrounds, which always

[1] The construction of the SCFT for $X_7 = N^{010}$ is in preparation [18].

[2] See however [12] where a similar approach for Q^{111} was attempted without, however, providing a match with Kaluza Klein spectra. Another attempt in this direction was also given in [66].

have exact marginal directions labeled by the coupling constants (the type IIB dilaton is a free parameter of the supergravity solution), these three dimensional fixed points may also be isolated. The only universal parameter in M-theory compactifications is l_p, which is related to the number of colours N, that is also the number of M2-branes (see chapter 1). The $1/N$ expansion in the gauge theory corresponds to the R_{AdS}/l_p expansion of M-theory through the relation $R_{AdS}/l_p \sim N^{1/6}$ (1.1.24), [1]. For large N, the M-theory solution is weakly coupled and supergravity can be used for studying the gauge theory.

The relevant degrees of freedom at the superconformal fixed points are in general different from the elementary fields of the supersymmetric gauge theory. For example, vector multiplets are not conformal in three dimensions and they should be replaced by some other multiplets of the superconformal group by dualizing the vector field to a scalar. Let us again consider the simple example of $\mathcal{N}=8$. The degrees of freedom at the superconformal point are contained in a supermultiplet with eight real scalars and eight fermions, transforming in representations of the global R-symmetry $SO(8)$. This is the same content of the $\mathcal{N}=8$ vector multiplet, when the vector field is dualized into a scalar. The change of variable from a vector to a scalar, which is well-defined in an abelian theory, is obviously a non-trivial and not even well-defined operation in a non-abelian theory. The scalars degrees of freedom at the superconformal point parametrize the flat space transverse to the M2-branes. In this case, the moduli space of vacua of the abelian $\mathcal{N}=8$ gauge theory, corresponding to a single M2-brane, is isomorphic to the transverse space. The case with N M2-branes is obtained by promoting the theory to a non-abelian one. We want to follow a similar procedure for the conifold cases.

For branes at the conifold singularity of $\mathcal{C}(X_7)$ there is no obvious way of reducing the system to a simple configuration of D2-branes in type IIA and read the field content by using standard brane techniques [3]. We can nevertheless use the intuition from geometry for identifying the relevant degrees of freedom at the superconformal point. We need an abelian gauge theory whose moduli space of vacua is isomorphic to $\mathcal{C}(X_7)$. The moduli space of vacua of $\mathcal{N}=2$ theories have two different branches touching at a point, the Coulomb branch parametrized by the *vev* of the scalars in the vector multiplets and the Higgs branch parametrized by the *vev* of the scalars in the chiral multiplets. The Higgs branch is the one we are interested in. Each of the two branches excludes the other, so we can consistently set the scalars in the vector multiplets to zero (see section 4.3.3 for a discussion of the scalar potential). We can find what we need in toric geometry. Indeed, this latter describes certain complex manifolds as Kähler quotients [71] associated to symplectic actions of a product of $U(1)$'s on some $\mathbb{C}^p$. This is completely equivalent to imposing the D-term equations for an abelian $\mathcal{N}=2, D=3$ gauge theory and dividing by the gauge group or, in other words, to finding the moduli space of vacua of the theory. Fortunately, both the cone over Q^{111} and that over M^{111} have a toric geometry description. This description was already used for studying these spaces in [12], [66]. Here, we will consider a different point of view. We can then easily find abelian gauge theories whose moduli space of vacua (the Higgs branch component) is isomorphic to these two particular conifolds. These abelian gauge theories will be then promoted to non abelian ones, whose IR fixed point will be our candidates as AdS/CFT–duals to the supergravities developed in chapter 3. We will find strong arguments that these theories are actually dual, giving in this way a non–trivial check of AdS/CFT correspondence.

[3] This possibility exists for orbifold singularities and was exploited in [67], [68], [69] for $\mathcal{N}=4$ and in [70] for $\mathcal{N}=2$.

A comment on the nomenclature. Most authors call the fundamental superfields of the gauge theory *supersingletons*. Actually this denomination is misleading, because they do not belong to the supersingleton representation of $Osp\,(\mathcal{N}|4)$ (see tables 2.11, 2.16 for the $\mathcal{N}=2$, $\mathcal{N}=3$ cases), but, in the case of $\mathcal{N}=2$, are chiral supermultiplets (see table 2.7). However, they are non unitary representations of the supergroup: in our case as we will show they have $E_0 = |y_0| < 1/2$; this is not a problem, because these superfields are degrees of freedom of the gauge theory, which does not have the $Osp\,(2|4)$ isometry, while the fundamental fields of the conformal theory are composite superfields, sitting in $Osp\,(2|4)$ UIRs. There are two reasons by which most authors call supersingletons the fundamental fields of the gauge theory; the first is that in the case $X_7 = S^7$ this is true; the second is an analogy: as the supersingleton superfields, they cannot be degrees of freedom of the conformal theory, while the composite superfields made by them can.

In section 1 I build, using rheonomy formalism (for a review on rheonomy see [37]), the generic three dimensional $\mathcal{N}=2$ supersymmetric gauge theory, writing the lagrangian and fixing the basis for the discussion of the next sections; furthermore, I write the $\mathcal{N}=4$ and $\mathcal{N}=8$ theories as $\mathcal{N}=2$ theories with constraints on the field content and the representations. In section 2 I discuss the geometry of the two manifolds Q^{111}, M^{111} as fiber bundles and as toric manifolds, and show how to find the abelian gauge theories associated with these toric descriptions. In section 3 I generalize these abelian gauge theories to non abelian ones, which are our candidates to be the AdS/CFT–duals to supergravity on $AdS_4 \times Q^{111}$ and $AdS_4 \times M^{111}$. I show that these theories perfectly reproduce the complete spectrum of shortened supergravity multiplets found in chapter 3. In section 4 I address the issue of the so–called baryonic operators, and show that they correspond to non–perturbative states of supergravity. They allow us to find the conformal weights of the fundamental fields of the gauge theory. In section 5 I draw the conclusions.

The content of the present chapter refers to results obtained within the collaborations [16], [15].

4.1 $\mathcal{N}=2$ three dimensional gauge theories and their rheonomic construction

As a first step, we construct a generic $\mathcal{N}=2$ gauge theory with an arbitrary gauge group and an arbitrary number of chiral multiplets in generic interaction. We are mostly interested in the final formulae for the scalar potential, which will be used in section 4.3.3, but we provide a complete construction of the lagrangian and of the supersymmetry transformation rules. To this effect we utilize the method of rheonomy [37] that yields the result for the lagrangian and the supersymmetry rules in component form avoiding the too much implicit notation of superfield formulation. Furthermore, we study the restrictions that guarantee an enlargement of supersymmetry to $\mathcal{N}=4$ or $\mathcal{N}=8$; in fact, even if in the case of M^{111} and Q^{111} the conformal theories do not seem to arise from deformation of more supersymmetric theories (as in [13]), in other cases this phenomenom could occur.

The first step in the rheonomic construction of a rigid supersymmetric theory involves writing the structural equations of rigid superspace. Then, we have to solve them in terms of the rheonomic expansion of the curvatures. Finally, we will write the superspace lagrangian in term of these curvatures, and, projecting this lagrangian on the bosonic

three–dimensional surface $\mathcal{M}_3$ we find the space–time lagrangian. This is the gauge theory lagrangian; we can introduce the YM coupling constant by scaling some or all of the scalar fields (actually in our theory we rescale all of them) by

$$z^i \longrightarrow g_{YM} z^i, \tag{4.1.1}$$

and multiplying the entire lagrangian by $1/g_{YM}^2$. The conformal IR fixed point is retrieved sending

$$g_{YM} \longrightarrow 0. \tag{4.1.2}$$

4.1.1 $\mathcal{N}=2$, $d=3$ rigid superspace

The d=3, $\mathcal{N}$–extended superspace is viewed as the supercoset space:

$$\mathcal{M}_{3|\mathcal{N}} = \frac{ISO(1,2|\mathcal{N})}{SO(1,2)} \equiv \frac{Z\left[ISO(1,2|\mathcal{N})\right]}{SO(1,2)\times \mathrm{I\!R}^{\mathcal{N}(\mathcal{N}-1)/2}} \tag{4.1.3}$$

where $ISO(1,2|\mathcal{N})$ (see section 2.5) is the $\mathcal{N}$–extended Poincaré superalgebra in three dimensions. It is the subalgebra of $Osp(\mathcal{N}|4)$ (see eq. (2.1.26)) spanned by the generators J_m, P_m, q^i. The central extension $Z\left[ISO(1,2|\mathcal{N})\right]$ which is not contained in $Osp(\mathcal{N}|4)$ is obtained by adjoining to $ISO(1,2|\mathcal{N})$ the central charges that generate the subalgebra $\mathrm{I\!R}^{\mathcal{N}(\mathcal{N}-1)/2}$. Specializing our analysis to the case $\mathcal{N}$=2, we can define the new generators:

$$\left\{ \begin{array}{rcl} Q & = & q^+ = \frac{1}{\sqrt{2}}(q^1 - iq^2) \\ Q^c & = & iq^- = \frac{1}{\sqrt{2}}(iq^1 - q^2) \\ Z & = & Z^{12} \end{array} \right. . \tag{4.1.4}$$

Before going on, I have to clarify the notations. In doing this computation, the conventions for two–component spinors are slightly modified with respect to the ones of chapter 2, in order to simplify the notations and avoid the explicit writing of spinor indices. The Grassman coordinates of $\mathcal{N}$=2 three-dimensional superspace introduced in equation (2.5.25) , $\theta^{\pm}_{\alpha}$, are renamed θ and θ^c. The reason for the superscript "^{c}" is that, in three dimensions the upper and lower components of the four–dimensional 4–component spinor are charge conjugate. In fact, the charge conjugation is defined by:

$$\theta^c \equiv C_{[3]}\overline{\theta}^T , \qquad \overline{\theta} \equiv \theta^\dagger \gamma^0 , \tag{4.1.5}$$

where $C_{[3]}$ is the $d=3$ charge conjugation matrix:

$$\left\{ \begin{array}{rcl} C_{[3]}\gamma^m C_{[3]}^{-1} & = & -(\gamma^m)^T \\ \gamma^0\gamma^m(\gamma^0)^{-1} & = & (\gamma^m)^\dagger . \end{array} \right. \tag{4.1.6}$$

The lower case gamma matrices are 2×2 and provide a realization of the d=2+1 Clifford algebra:

$$\{\gamma^m , \gamma^n\} = \eta^{mn} \tag{4.1.7}$$

Utilizing the following explicit basis:

$$\left\{ \begin{array}{rcl} \gamma^0 & = & \sigma^2 \\ \gamma^1 & = & -i\sigma^3 \\ \gamma^2 & = & -i\sigma^1 \end{array} \right. \qquad C_{[3]} = -i\sigma^2 , \tag{4.1.8}$$

both γ^0 and $C_{[3]}$ become proportional to $\varepsilon_{\alpha\beta}$. This implies that in equation (4.1.5) the role of the matrices $C_{[3]}$ and γ^0 is just to convert upper into lower $SL(2,\mathbb{C})$ indices and viceversa.

The relation between the two notations for the spinors is summarized in the following table:

$$\begin{array}{|c|c|} (\theta^+)^\alpha & \theta \\ (\theta^+)_\alpha & \overline{\theta}^c \\ (\theta^-)^\alpha & -i\theta^c \\ (\theta^-)_\alpha & -i\overline{\theta} \end{array} \tag{4.1.9}$$

With the second set of conventions the spinor indices can be ignored since the contractions are always made between barred (on the left) and unbarred (on the right) spinors.

The left invariant one–form Ω on $\mathcal{M}_{3|\mathcal{N}}$ is:

$$\Omega = V^m P_m - \tfrac{1}{2} i\omega^{mn} J_{mn} + \overline{\psi^c} Q + \overline{\psi} Q^c + iAZ\,. \tag{4.1.10}$$

The superalgebra (2.1.26) defines all the structure constants apart from those relative to the central charge that are trivially determined. Hence we can write:

$$\begin{aligned} d\Omega - \Omega\wedge\Omega \;\; = \;\; & \left(dV^m - \omega^m_{\ n}\wedge V^n + i\overline{\psi}\wedge\gamma^m\psi + i\overline{\psi}^c\wedge\gamma^m\psi^c\right)P_m + \\ & -\tfrac{1}{2} i\left(d\omega^{mn} - \omega^m_{\ p}\wedge\omega^{pn}\right)J_{mn} + \\ & + \left(d\overline{\psi}^c + \tfrac{1}{2} i\omega^{mn}\wedge\overline{\psi}^c\gamma_{mn}\right)Q + \\ & + \left(d\overline{\psi} - \tfrac{1}{2} i\omega^{mn}\wedge\overline{\psi}\gamma_{mn}\right)Q^c + \\ & + i\left(dA + i\overline{\psi}^c\wedge\psi^c - i\overline{\psi}\wedge\psi\right)Z\,. \end{aligned} \tag{4.1.11}$$

Imposing the Maurer-Cartan equation $d\Omega - \Omega\wedge\Omega = 0$ is equivalent to imposing flatness in superspace, i.e. global supersymmetry. So we have

$$\left\{\begin{array}{ccl} dV^m - \omega^m_{\ n}\wedge V^n & = & -i\overline{\psi}^c\wedge\gamma^m\psi^c - i\overline{\psi}\wedge\gamma^m\psi \\ d\omega^{mn} & = & \omega^m_{\ p}\wedge\omega^{pn} \\ d\overline{\psi} & = & \tfrac{1}{2}\omega^{mn}\wedge\overline{\psi}\gamma_{mn} \\ d\overline{\psi}^c & = & -\tfrac{1}{2}\omega^{mn}\wedge\overline{\psi}^c\gamma_{mn} \\ dA & = & -i\overline{\psi}^c\wedge\psi^c - i\overline{\psi}\wedge\psi\,. \end{array}\right. \tag{4.1.12}$$

The simplest solution for the supervielbein and connection is:

$$\left\{\begin{array}{ccl} V^m & = & dx^m - i\overline{\theta}^c\gamma^m d\theta^c - i\overline{\theta}\gamma^m d\theta \\ \omega^{mn} & = & 0 \\ \psi & = & d\theta \\ \psi^c & = & d\theta^c \\ A & = & -i\overline{\theta}^c\, d\theta^c - i\overline{\theta}\, d\theta\,. \end{array}\right. \tag{4.1.13}$$

The superderivatives discussed in section 2.5.2 (compare with eq.(2.5.27)),

$$\left\{\begin{array}{ccl} D_m & = & \partial_m \\ D & = & \frac{\partial}{\partial\overline{\theta}} - i\gamma^m\theta\partial_m \\ D^c & = & \frac{\partial}{\partial\overline{\theta}^c} - i\gamma^m\theta^c\partial_m \end{array}\right. , \tag{4.1.14}$$

are the vectors dual to these one–forms.

4.1.2 Rheonomic construction of the $\mathcal{N}=2,\ d=3$, lagrangian

As stated we are interested in the generic form of $\mathcal{N}=2,\ d=3$ super Yang Mills theory coupled to n chiral multiplets arranged into a generic representation $\mathcal{R}$ of the gauge group $\mathcal{G}$.

In $\mathcal{N}=2,\ d=3$ supersymmetric theories, two formulations are allowed: the on–shell and the off–shell one. In the on–shell formulation which contains only the physical fields, the supersymmetry transformations rules close the supersymmetry algebra only upon use of the field equations. On the other hand the off–shell formulation contains further auxiliary, non dynamical fields that make it possible for the supersymmetry transformations rules to close the supersymmetry algebra identically. By solving the field equations of the auxiliary fields these latter can be eliminated and the on–shell formulation can be retrieved. We adopt the off–shell formulation.

The gauge multiplet

The three–dimensional $\mathcal{N}=2$ vector multiplet contains the following Lie-algebra valued fields:

$$(\mathcal{A},\lambda,\lambda^c,M,P)\ , \tag{4.1.15}$$

where $\mathcal{A}=\mathcal{A}^I t_I$ is the real gauge connection one–form, λ and λ^c are two complex Dirac spinors (the *gauginos*), M and P are real scalars; P is an auxiliary field.

The field strength is:

$$F=d\mathcal{A}+i\mathcal{A}\wedge\mathcal{A}\,. \tag{4.1.16}$$

The covariant derivative on the other fields of the gauge multiplets is defined as:

$$\nabla X=dX+i\left[\mathcal{A},X\right]\,. \tag{4.1.17}$$

From (4.1.16) and (4.1.17) we obtain the Bianchi identity:

$$\nabla^2 X=i\left[F,X\right]\,. \tag{4.1.18}$$

The rheonomic parametrization of the *curvatures* is given by:

$$\left\{\begin{array}{rcl} F &=& F_{mn}V^mV^n-i\overline{\psi}^c\gamma_m\lambda V^m-i\overline{\psi}\gamma_m\lambda^cV^m+iM\left(\overline{\psi}\psi-\overline{\psi}^c\psi^c\right)\\ \nabla\lambda &=& V^m\nabla_m\lambda+\nabla\!\!\!\!/ M\psi^c-F_{mn}\gamma^{mn}\psi^c+iP\psi^c\\ \nabla\lambda^c &=& V^m\nabla_m\lambda^c-\nabla\!\!\!\!/ M\psi-F_{mn}\gamma^{mn}\psi-iP\psi\\ \nabla M &=& V^m\nabla_m M+i\overline{\psi}\lambda^c-i\overline{\psi}^c\lambda\\ \nabla P &=& V^m\nabla_m P+\overline{\psi}\nabla\!\!\!\!/\lambda^c-\overline{\psi}^c\nabla\!\!\!\!/\lambda-i\overline{\psi}\left[\lambda^c,M\right]-i\overline{\psi}^c\left[\lambda,M\right] \end{array}\right. \tag{4.1.19}$$

and we also have:

$$\left\{\begin{array}{rcl} \nabla F_{mn} &=& V^p\nabla_pF_{mn}+i\overline{\psi}^c\gamma_{[m}\nabla_{n]}\lambda+i\overline{\psi}\gamma_{[m}\nabla_{n]}\lambda^c\\ \nabla\nabla_m M &=& V^n\nabla_n\nabla_m M+i\overline{\psi}\nabla_m\lambda^c-i\overline{\psi}^c\nabla_m\lambda+\overline{\psi}^c\gamma_m\left[\lambda,M\right]+\overline{\psi}\gamma_m\left[\lambda^c,M\right]\\ \nabla\nabla_m\lambda &=& V^n\nabla_n\nabla_m\lambda+\nabla_m\nabla_nM\gamma^n\psi^c-\nabla_mF_{np}\gamma^{np}\psi^c+\\ && +i\nabla_mP\psi^c+\overline{\psi}\gamma_m\left[\lambda^c,\lambda\right]\\ \nabla_{[p}F_{mn]} &=& 0\\ \nabla_{[m}\nabla_{n]}M &=& i\left[F_{mn},M\right]\\ \nabla_{[m}\nabla_{n]}\lambda &=& i\left[F_{mn},\lambda\right]\,. \end{array}\right. \tag{4.1.20}$$

The off–shell formulation of the theory contains an arbitrariness in the choice of the functional dependence of the auxiliary fields on the physical fields. Consistency with the Bianchi identities forces the generic expression of P as a function of M to be:

$$P^I = 2\alpha M^I + \zeta^{\widetilde{I}} \mathcal{C}_{\widetilde{I}}{}^I \,, \tag{4.1.21}$$

where α, $\zeta^{\widetilde{I}}$ are arbitrary real parameters and $\mathcal{C}_{\widetilde{I}}{}^I$ is the projector on the center $Z[\mathcal{G}]$ of the gauge Lie algebra. The terms in the lagrangian proportional to α and ζ are separately supersymmetric. In the bosonic lagrangian, the part proportional to α is a Chern Simons term, while the part proportional to ζ constitutes the Fayet Iliopoulos term. Note that the Fayet Iliopoulos terms are associated only with a central abelian subalgebra of the gauge algebra $\mathcal{G}$.

Enforcing (4.1.21) we get the following equations of motion for the spinors:

$$\begin{cases} \nabla\!\!\!\!/\,\lambda = 2i\alpha\lambda - i\,[\lambda, M] \\ \\ \nabla\!\!\!\!/\,\lambda^c = 2i\alpha\lambda^c + i\,[\lambda^c, M] \;. \end{cases} \tag{4.1.22}$$

Taking the covariant derivatives of these, we obtain the equations of motion for the bosonic fields:

$$\begin{cases} \nabla_m \nabla^m M = -4\alpha^2 M - 2\alpha\beta - 2\left[\overline{\lambda}, \lambda\right] \\ \nabla^n F_{mn} = -\alpha\epsilon_{mnp} F^{np} - \frac{i}{2}\left[\nabla_m M, M\right] \,. \end{cases} \tag{4.1.23}$$

Using the rheonomic approach we find the following superspace lagrangian for the gauge multiplet:

$$\mathcal{L}_{gauge} = \mathcal{L}^{Maxwell}_{gauge} + \mathcal{L}^{Chern-Simons}_{gauge} + \mathcal{L}^{Fayet-Iliopoulos}_{gauge} \,, \tag{4.1.24}$$

where

$$\begin{aligned} \mathcal{L}^{Maxwell}_{gauge} &= Tr\Big\{-2F^{mn}\left[F + i\overline{\psi}^c\gamma_m\lambda V^m + i\overline{\psi}\gamma_m\lambda^c V^m - 2iM\overline{\psi}\psi\right] V^p\epsilon_{mnp} + \\ &+ \tfrac{1}{3}F_{qr}F^{qr}V^mV^nV^p\epsilon_{mnp} - \tfrac{1}{2}i\epsilon_{mnp}\left[\nabla\overline{\lambda}\gamma^m\lambda + \nabla\overline{\lambda}^c\gamma^m\lambda^c\right]V^nV^p + \\ &+ \epsilon_{mnp}\mathcal{M}^m\left[\nabla M - i\overline{\psi}\lambda^c + i\overline{\psi}^c\lambda\right]V^nV^p - \tfrac{1}{6}\mathcal{M}^d\mathcal{M}_d\epsilon_{mnp}V^mV^nV^p + \\ &+ 2\nabla M\overline{\psi}^c\gamma_c\lambda V^p - 2\nabla M\overline{\psi}\gamma_p\lambda^c V^p + \\ &+ 2F\overline{\psi}^c\lambda + 2F\overline{\psi}\lambda^c + i\overline{\lambda}^c\lambda\overline{\psi}^c\gamma_m\psi V^m + i\overline{\lambda}\lambda^c\overline{\psi}\gamma_m\psi^c V^m + \\ &+ \tfrac{1}{6}\mathcal{P}^2V^mV^nV^p\epsilon_{mnp} - 4i(\overline{\psi}\psi)M\left[\overline{\psi}^c\lambda + \overline{\psi}\lambda^c\right]\Big\} \,, \end{aligned} \tag{4.1.25}$$

$$\begin{aligned} \mathcal{L}^{Chern-Simons}_{gauge} &= \alpha Tr\left\{-2\left(\mathcal{A}\wedge F - i\mathcal{A}\wedge\mathcal{A}\wedge\chi A\right) - \tfrac{2}{3}MP\epsilon_{mnp}V^mV^nV^p + \right. \\ &+ \tfrac{2}{3}\overline{\lambda}\lambda\epsilon_{mnp}V^mV^nV^p + 2M\epsilon_{mnp}\left[\overline{\psi}^c\gamma^m\lambda + \overline{\psi}\gamma^m\lambda^c\right]V^nV^p + \\ &\left. -4iM^2\overline{\psi}\gamma_m\psi V^m\right\} \end{aligned} \tag{4.1.26}$$

$$\begin{aligned} \mathcal{L}^{Fayet-Iliopoulos}_{gauge} &= Tr\Big\{\zeta\mathcal{C}\Big[-\tfrac{1}{3}P\epsilon_{mnp}V^mV^nV^p\epsilon_{mnp}\left(\overline{\psi}^c\gamma^m\lambda - \overline{\psi}\gamma^m\lambda^c\right)V^nV^p + \\ &-4iM\overline{\psi}\gamma_m\psi V^m - 4i\mathcal{A}\overline{\psi}\psi\Big]\Big\} \,. \end{aligned} \tag{4.1.27}$$

Chiral multiplet

The chiral multiplet contains the following fields:

$$\left(z^i, \chi^i, H^i\right) \tag{4.1.28}$$

where z^i are complex scalar fields which parametrize a Kähler manifold. Since we are interested in microscopic theories with canonical kinetic terms we take this Kähler manifold to be flat and we choose its metric to be the constant $\eta_{ij^*} \equiv \mathrm{diag}(+,+,\ldots,+)$. The other fields in the chiral multiplet are χ^i which is a two components Dirac spinor and H^i which is a complex scalar auxiliary field. The index i runs in the representation $\mathcal{R}$ of $\mathcal{G}$.

The covariant derivative of the fields X^i in the chiral multiplet is:

$$\nabla X^i = dX^i + i\eta^{ii^*}\mathcal{A}^I (T_I)_{i^*j} X^j\,, \tag{4.1.29}$$

where $(T_I)_{i^*j}$ are the hermitian generators of $\mathcal{G}$ in the representation $\mathcal{R}$. The covariant derivative of the complex conjugate fields $\overline{X}^{i^*}$ is:

$$\nabla \overline{X}^{i^*} = d\overline{X}^{i^*} - i\eta^{i^*i}\mathcal{A}^I (\overline{T}_I)_{ij^*} \overline{X}^{j^*}\,, \tag{4.1.30}$$

where

$$(\overline{T}_I)_{ij^*} \equiv \overline{(T_I)_{i^*j}} = (T_I)_{j^*i}\,. \tag{4.1.31}$$

The rheonomic parametrization of the curvatures is given by:

$$\left\{ \begin{array}{rcl} \nabla z^i &=& V^m \nabla_m z^i + 2\overline{\psi}^c \chi^i \\ \nabla \chi^i &=& V^m \nabla_m \chi^i - i\nabla\!\!\!\!/ z^i \psi^c + H^i \psi - M^I (T_I)^i{}_j z^j \psi^c \\ \nabla H^i &=& V^m \nabla_m H^i - 2i\overline{\psi}\nabla\!\!\!\!/ \chi^i - 2i\overline{\psi}\lambda^I (T_I)^i{}_j z^j + 2M^I (T_I)^i{}_j \overline{\psi}\chi^j \end{array} \right. . \tag{4.1.32}$$

We can choose the auxiliary fields H^i to be the derivatives of an arbitrary antiholomorphic superpotential $\overline{W}(\overline{z})$:

$$H^i = \eta^{ij^*} \frac{\partial \overline{W}(\overline{z})}{\partial z^{j^*}} = \eta^{ij^*} \partial_{j^*} \overline{W}\,. \tag{4.1.33}$$

Enforcing eq. (4.1.33) we get the following equations of motion for the spinors:

$$\left\{ \begin{array}{l} \nabla\!\!\!\!/ \chi^i = i\eta^{ij^*}\partial_{j^*}\partial_{k^*}\overline{W}\chi^{ck^*} - \lambda^I (T_I)^i{}_j z^j - iM^I (T_I)^i{}_j \chi^j \\ \\ \nabla\!\!\!\!/ \chi^{ci^*} = i\eta^{i^*j}\partial_j\partial_k W \chi^k + \lambda^{cI} (\overline{T}_I)^{i^*}_{j^*} \overline{z}^{j^*} - iM^I (\overline{T}_I)^{i^*}_{j^*} \chi^{cj^*} \end{array} \right. . \tag{4.1.34}$$

Taking the differential of (4.1.34) one obtains the equation of motion for z:

$$\begin{array}{rcl} \Box z^i &=& \eta^{ii^*}\partial_{i^*}\partial_{j^*}\partial k^* \overline{W}\left(\overline{\chi}^{j^*}\chi^{ck^*}\right) - \eta^{ij^*}\partial_{j^*}\partial_{k^*}\overline{W}(\overline{z})\partial_i W + \\ && +P^I (T_I)^i{}_j z^j - M^I M^J (T_I T_J)^i{}_j z^j - 2i\overline{\lambda}^I (T_I)^i{}_j \chi^j\,. \end{array} \tag{4.1.35}$$

The first order Lagrangian for the chiral multiplet (4.1.28) is:

$$\mathcal{L}_{chiral} = \mathcal{L}_{chiral}^{Wess-Zumino} + \mathcal{L}_{chiral}^{superpotential}\,, \tag{4.1.36}$$

where

$$
\begin{aligned}
\mathcal{L}^{Wess-Zumino}_{chiral} &= \epsilon_{mnp}\overline{\Pi}^{m\,i^*}\eta_{i^*j}\left[\nabla z^j - 2\overline{\psi}^c\chi^j\right]V^nV^p + \\
&+ \epsilon_{mnp}\Pi^{m\,i}\eta_{ij^*}\left[\nabla\overline{z}^{j^*} - 2\overline{\chi}\psi^{c\,j^*}\right]V^nV^p + \\
&- \tfrac{1}{3}\epsilon_{mnp}\eta_{ij^*}\Pi^i_q\overline{\Pi}^{q\,j^*}V^mV^nV^p + \\
&+ i\epsilon_{mnp}\eta_{ij^*}\left[\overline{\chi}^{j^*}\gamma^m\nabla\chi^i + \overline{\chi}^{c\,i}\gamma^m\nabla\chi^{c\,j^*}\right]V^nV^p + \\
&+ 4i\eta_{ij^*}\left[\nabla z^i\overline{\psi}\gamma_m\chi^{c\,j^*} - \nabla\overline{z}^{j^*}\overline{\chi}^{c\,i}\gamma_m\psi\right]V^m + \\
&- 4i\eta_{ij^*}\left(\overline{\chi}^{j^*}\gamma_m\chi^i\right)\left(\overline{\psi}^c\psi^c\right)V^m - 4i\eta_{ij^*}\left(\overline{\chi}^{j^*}\chi^i\right)\left(\overline{\psi}^c\gamma_m\psi^c\right)V^m + \\
&+ \tfrac{1}{3}\eta_{ij^*}H^i\overline{H}^{j^*}\epsilon_{mnp}V^mV^nV^p + 2\left(\overline{\psi}\psi\right)\eta_{ij^*}\left[\overline{z}^{j^*}\nabla z^i - z^i\nabla\overline{z}^{j^*}\right] + \\
&+ 2i\epsilon_{mnp}z^iM^I(T_I)_{ij^*}\overline{\chi}^{j^*}\gamma^m\psi^cV^nV^p + \\
&+ 2i\epsilon_{mnp}\overline{z}^{j^*}M^I(T_I)_{j^*i}\overline{\chi}^{c\,i}\gamma^m\psi V^nV^p + \\
&- \tfrac{2}{3}M^I(T_I)_{ij^*}\overline{\chi}^{j^*}\chi^i\epsilon_{mnp}V^mV^nV^p + \\
&+ \tfrac{2}{3}i\left[\overline{\chi}^{j^*}\lambda^I(T_I)_{j^*i}z^i - \overline{\chi}^{c\,i}\lambda^{c\,I}(T_I)_{ij^*}\overline{z}^{j^*}\right]\epsilon_{mnp}V^mV^nV^p + \\
&+ \tfrac{1}{3}z^iP^I(T_I)_{ij^*}\overline{z}^{j^*}\epsilon_{mnp}V^mV^nV^p + \\
&- \left(\overline{\psi}^c\gamma^m\lambda^I(T_I)_{ij^*}\right)z^i\overline{z}^{j^*}\epsilon_{mnp}V^nV^p + \\
&+ \left(\overline{\psi}\gamma^m\lambda^{c\,I}(T_I)_{ij^*}\right)z^i\overline{z}^{j^*}\epsilon_{mnp}V^nV^p + \\
&- \tfrac{1}{3}M^IM^J\,z^i(T_IT_J)_{ij^*}\overline{z}^{j^*}\epsilon_{mnp}V^mV^nV^p + \\
&+ 4iM^I(T_I)_{ij^*}z^i\overline{z}^{j^*}\overline{\psi}\gamma_m\psi V^m\,,
\end{aligned}
\tag{4.1.37}
$$

and

$$
\begin{aligned}
\mathcal{L}^{superpotential}_{chiral} &= -2i\epsilon_{mnp}\left[\overline{\chi}^{j^*}\gamma^m\partial_{j^*}\overline{W}(\overline{z})\psi + \overline{\chi}^{c\,j}\gamma^m\partial_j\overline{W}(z)\psi^c\right]V^nV^p + \\
&+ \tfrac{1}{3}\left[\partial_i\partial_jW(z)\overline{\chi}^{c\,i}\chi^j + \partial_{i^*}\partial_{j^*}\overline{W}(\overline{z})\overline{\chi}^{i^*}\chi^{c\,j^*}\right]\epsilon_{mnp}V^mV^nV^p + \\
&- \tfrac{1}{3}\left[H^i\partial_iW(z) + \overline{H}^{j^*}\partial_{j^*}\overline{W}(\overline{z}) - \eta_{ij^*}H^i\overline{H}^{j^*}\right]\epsilon_{mnp}V^mV^nV^p + \\
&- 4i\left[W(z) + \overline{W}(\overline{z})\right]\overline{\psi}\gamma_m\psi^cV^m\,.
\end{aligned}
\tag{4.1.38}
$$

The space–time Lagrangian

In the rheonomic approach ([37]), the total three–dimensional $\mathcal{N}$=2 lagrangian:

$$\mathcal{L}^{\mathcal{N}=2} = \mathcal{L}_{gauge} + \mathcal{L}_{chiral} \tag{4.1.39}$$

is a closed ($d\mathcal{L}^{\mathcal{N}=2} = 0$) three–form defined in superspace. The action is given by the integral of $\mathcal{L}^{\mathcal{N}=2}$ on a generic *bosonic* three–dimensional surface $\mathcal{M}_3$ in superspace:

$$S = \int_{\mathcal{M}_3}\mathcal{L}^{\mathcal{N}=2}\,. \tag{4.1.40}$$

Supersymmetry transformations can be viewed as global translations in superspace which move $\mathcal{M}_3$. Then, being $\mathcal{L}^{\mathcal{N}=2}$ closed, the action is invariant under global supersymmetry transformations.

We choose as bosonic surface the one defined by:

$$\theta = d\theta = 0\,. \tag{4.1.41}$$

Then the space–time lagrangian, i.e. the pull–back of $\mathcal{L}^{\mathcal{N}=2}$ on $\mathcal{M}_3$, is:

$$\mathcal{L}^{\mathcal{N}=2}_{st} = \mathcal{L}^{kinetic}_{st} + \mathcal{L}^{fermion\ mass}_{st} + \mathcal{L}^{potential}_{st}\,, \tag{4.1.42}$$

where

$$\begin{aligned}
\mathcal{L}^{kinetic}_{st} &= \Big\{\eta_{ij^*}\nabla_m z^i \nabla^m \overline{z}^{j^*} + i\eta_{ij^*}\left(\overline{\chi}^{j^*}\nabla\!\!\!\!/\,\chi^i + \overline{\chi}^{c\,i}\nabla\!\!\!\!/\,\chi^{c\,j^*}\right) + \\
&\quad -g_{IJ}F^I_{mn}F^{J\,mn} + \tfrac{1}{2}g_{IJ}\nabla_m M^I \nabla^m M^J + \\
&\quad + \tfrac{1}{2}ig_{IJ}\left(\overline{\lambda}^I\nabla\!\!\!\!/\,\lambda^J + \overline{\lambda}^{c\,I}\nabla\!\!\!\!/\,\lambda^{c\,J}\right)\Big\}\,d^3x
\end{aligned} \tag{4.1.43}$$

$$\begin{aligned}
\mathcal{L}^{fermion\ mass}_{st} &= \Big\{i\left(\overline{\chi}^{c\,i}\partial_i\partial_j W(z)\chi^j + \overline{\chi}^{i^*}\partial_{i^*}\partial_{j^*}\overline{W}(\overline{z})\chi^{c\,j^*}\right) + \\
&\quad -f_{IJK}M^I\overline{\lambda}^J\lambda^K - 2\overline{\chi}^{i^*}M^I(T_I)_{ij^*}\chi^{j^*} + \\
&\quad +2i\left(\overline{\chi}^{i^*}\lambda^I(T_I)_{i^*j}z^j - \overline{\chi}^{c\,i}\lambda^I(T_I)_{ij^*}\overline{z}^{j^*}\right) + \\
&\quad +2\alpha g_{IJ}\overline{\lambda}^I\lambda^J\Big\}\,d^3x
\end{aligned} \tag{4.1.44}$$

$$\mathcal{L}^{potential}_{st} = -U(z,\overline{z},H,\overline{H},M,P)d^3x\,, \tag{4.1.45}$$

and

$$\begin{aligned}
U(z,\overline{z},H,\overline{H},M,P) &= H^i\partial_i W(z) + \overline{H}^{j^*}\partial_{j^*}\overline{W}(\overline{z}) - \eta_{ij^*}H^i\overline{H}^{j^*} + \\
&\quad -\tfrac{1}{2}g_{IJ}P^IP^J - z^iP^I(T_I)_{ij^*}\overline{z}^{j^*} + \\
&\quad +z^iM^I(T_I)_{ij^*}\eta^{j^*k}M^J(T_J)_{kl^*}\overline{z}^{l^*} + \\
&\quad +2\alpha g_{IJ}M^IP^J + \zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\ I}g_{IJ}P^J\,.
\end{aligned} \tag{4.1.46}$$

From the variation of the lagrangian with respect to the auxiliary fields H^i and P^I we find:

$$\begin{aligned}
H^i &= \eta^{ij^*}\partial_{j^*}\overline{W}(\overline{z})\,, && (4.1.47)\\
P^I &= D^I(z,\overline{z}) + 2\alpha M^I + \zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\ I}\,, && (4.1.48)
\end{aligned}$$

where

$$D^I(z,\overline{z}) = -\overline{z}^{i^*}(T_I)_{i^*j}z^j\,. \tag{4.1.49}$$

Substituting this expression in the potential (4.1.46) we obtain:

$$\begin{aligned}
U(z,\overline{z},M) &= \partial_i W(z)\eta^{ij^*}\partial_{j^*}\overline{W}(\overline{z}) + \\
&\quad +\tfrac{1}{2}g^{IJ}\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right)\left(\overline{z}^{k^*}(T_J)_{k^*l}z^l\right) + \\
&\quad +\overline{z}^{i^*}M^I(T_I)_{i^*j}\eta^{jk^*}M^J(T_J)_{k^*l}z^l + \\
&\quad +2\alpha^2 g_{IJ}M^IM^J + 2\alpha\zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\ I}g_{IJ}M^J + \tfrac{1}{2}\zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\ I}g_{IJ}\zeta^{\widetilde{J}}\mathcal{C}_{\widetilde{J}}^{\ J} + \\
&\quad -2\alpha M^I\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right) - \zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\ I}\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right)\,.
\end{aligned} \tag{4.1.50}$$

4.1.3 A particular $\mathcal{N}=2$ theory: $\mathcal{N}=4$

A general lagrangian for matter coupled rigid $\mathcal{N}=4, d=3$ super Yang Mills theory is easily obtained from the dimensional reduction of the $\mathcal{N}=2, d=4$ gauge theory (see [72]). The bosonic sector of this latter lagrangian is the following:

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=4}_{bosonic} &= -\frac{1}{g^2_{YM}} g_{IJ} F^I_{mn} F^{J\,mn} + \tfrac{1}{2g^2_{YM}} g_{IJ} \nabla_m M^I \nabla^m M^J + \\
&\quad + \frac{2}{g^2_{YM}} g_{IJ} \nabla_m \overline{Y}^I \nabla^m Y^J + \tfrac{1}{2} Tr\left(\nabla_m \overline{\mathbf{Q}} \nabla^m \mathbf{Q}\right) + \\
&\quad - \frac{1}{g^2_{YM}} g_{IN} f^I_{JK} f^N_{LM}\, M^J \overline{Y}^K\, M^L Y^M - M^I M^J Tr\left(\overline{\mathbf{Q}}(T_I\, T_J)\mathbf{Q}\right) + \\
&\quad - \frac{2}{g^2_{YM}} g_{IN} f^I_{JK} f^N_{LM}\, \overline{Y}^J Y^K\, \overline{Y}^L Y^M - \overline{Y}^I Y^J\, Tr\left(\overline{\mathbf{Q}}\,\{T_I, T_J\}\,\mathbf{Q}\right) + \\
&\quad - \tfrac{1}{4} g^2_{YM} g_{IJ} Tr\left(\overline{\mathbf{Q}}(T^I)\mathbf{Q}\,\overline{\mathbf{Q}}(T^J)\mathbf{Q}\right) .
\end{aligned} \tag{4.1.51}$$

The bosonic matter field content is given by two kinds of fields. First we have a complex field Y^I in the adjoint representation of the gauge group, which belongs to a chiral multiplet. Secondly, we have an n-uplet of quaternions $\mathbf{Q}$, which parametrize a (flat)[4] HyperKähler manifold:

$$\mathbf{Q} = \begin{pmatrix} Q^1 & = & q^{1|0}\mathbb{1} - i q^{1|x}\sigma_x \\ Q^2 & = & q^{2|0}\mathbb{1} - i q^{2|x}\sigma_x \\ & \dots & \\ Q^A & = & q^{A|0}\mathbb{1} - i q^{A|x}\sigma_x \\ & \dots & \\ Q^n & = & q^{n|0}\mathbb{1} - i q^{n|x}\sigma_x \end{pmatrix} . \qquad \begin{array}{l} q^{A|0}, q^{A|x} \in \mathbb{R} \\ \\ A \in \{1, \dots, n\} \\ \\ x \in \{1,2,3\} \end{array} \tag{4.1.52}$$

The quaternionic conjugation is defined by:

$$\overline{Q}^A = q^{A|0}\mathbb{1} + i q^{A|x}\sigma_x \,. \tag{4.1.53}$$

In this realization, the quaternions are represented by matrices of the form:

$$Q^A = \begin{pmatrix} u^A & i\overline{v}_{A^*} \\ iv_A & \overline{u}^{A^*} \end{pmatrix} \qquad \overline{Q}^A = \begin{pmatrix} \overline{u}^{A^*} & -i\overline{v}_{A^*} \\ -iv_A & u^A \end{pmatrix} \qquad \begin{array}{l} u^A = q^{A|0} - i q^{A|3} \\ v^A = -q^{A|1} - i q^{A|2} \,. \end{array} \tag{4.1.54}$$

The generators of the gauge group $\mathcal{G}$ have a triholomorphic action on the flat HyperKähler manifold, namely they respect the three complex structures [71]. Explicitly this triholomorphic action on $\mathbf{Q}$ is the following:

$$\begin{aligned}
\delta^I \mathbf{Q} &= i \hat{T}^I \mathbf{Q} \\
\delta^I \begin{pmatrix} u^A & i\overline{v}_{A^*} \\ iv_A & \overline{u}^{A^*} \end{pmatrix} &= i \begin{pmatrix} T^I_{A^*B} & \\ & -\overline{T}^I_{AB^*} \end{pmatrix} \begin{pmatrix} u^B & i\overline{v}_{B^*} \\ iv_B & \overline{u}^{B^*} \end{pmatrix}
\end{aligned} \tag{4.1.55}$$

[4] Once again we choose the HyperKähler manifold to be flat since we are interested in microscopic theories with canonical kinetic terms

where the $T^I_{A^*B}$ realize a representation of $\mathcal{G}$ in terms of $n \times n$ hermitian matrices. We define $\overline{T}_{AB^*} \equiv (T_{A^*B})^*$, so, being the generators hermitian ($T^* = T^T$), we can write:

$$T_{A^*B} = \overline{T}_{BA^*}. \tag{4.1.56}$$

We can rewrite eq. (4.1.51) in the form:

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=4}_{bosonic} \;=\; & -\frac{1}{g^2_{_{YM}}} g_{IJ} F^I_{mn} F^{J\,mn} + \frac{1}{2g^2_{_{YM}}} g_{IJ} \nabla_m M^I \nabla^m M^J + \\
& + \frac{2}{g^2_{_{YM}}} \gamma_{IJ} \nabla_m \overline{Y}^I \nabla^m Y^J + \nabla_m \overline{u} \nabla^m u + \nabla_m \overline{v} \nabla^m v + \\
& - \frac{2}{g^2_{_{YM}}} M^I M^J \overline{Y}^R f_{RIL} f^L_{JS} Y^S - M^I M^J \left(\overline{u} T_I T_J u + \overline{v} \overline{T}_I \overline{T}_J v \right) + \\
& - \frac{2}{g^2_{_{YM}}} g_{IJ} \left[\overline{Y}, Y \right]^I \left[\overline{Y}, Y \right]^J - 2 \overline{Y}^I Y^J \left(\overline{u} \{ T_I, T_J \} u + \overline{v} \{ \overline{T}_I, \overline{T}_J \} v \right) + \\
& - 2 g^2_{_{YM}} g_{IJ} \left(v T^I u \right) \left(\overline{v} \overline{T}^J \overline{u} \right) - \tfrac{1}{2} g^2_{_{YM}} g_{IJ} \Big[\left(\overline{u} T^I u \right) \left(\overline{u} T^J u \right) + \\
& + (\overline{v} \overline{T}^I v)(\overline{v} \overline{T}^J v) - 2 (\overline{u} T^I u)(\overline{v} \overline{T}^J v) \Big] \,.
\end{aligned} \tag{4.1.57}$$

By comparing the bosonic part of (4.1.42) with (4.1.57), we see that in order for a $\mathcal{N}$=2 lagrangian to be also $\mathcal{N}$=4 supersymmetric, the matter content of the theory and the form of the superpotential are constrained. The chiral multiplets have to be in an adjoint plus a generic quaternionic representation of $\mathcal{G}$. So the fields z^i and the gauge generators are

$$z^i = \begin{cases} \sqrt{2} Y^I \\ g_{YM} u^A \\ g_{YM} v_A \end{cases} \qquad T^I_{i^*j} = \begin{cases} f^I_{JK} \\ (T^I)_{A^*B} \\ -(\overline{T}^I)_{AB^*} \end{cases} \,. \tag{4.1.58}$$

Moreover, the holomorphic superpotential $W(z)$ has to be of the form:

$$W\left(Y, u, v\right) = 2 g^4_{_{YM}} \delta^{AA^*} Y^I \, v_A (T_I)_{A^*B} u^B \,. \tag{4.1.59}$$

Substituting these choices in the supersymmetric lagrangian (4.1.42) we obtain the general $\mathcal{N}$=4 lagrangian expressed in $\mathcal{N}$=2 language.

Since the action of the gauge group is triholomorphic there is a triholomorphic momentum map associated with each gauge group generator (see [73], [74], [72]).

The momentum map is given by:

$$\mathcal{P} = \tfrac{1}{2} \left(\overline{\mathbf{Q}} \, \hat{T} \, \mathbf{Q} \right) = \begin{pmatrix} \mathcal{P}_3 & \mathcal{P}_+ \\ \mathcal{P}_- & -\mathcal{P}_3 \end{pmatrix} \,, \tag{4.1.60}$$

where

$$\begin{aligned}
\mathcal{P}^I_3 &= i \left(\overline{u} T^I u - \overline{v} \overline{T}^I v \right) = -i D^I \\
\mathcal{P}^I_+ &= 2 i \overline{v} \overline{T}^I \overline{u} = i g^{-4}_{_{YM}} \; \partial \overline{W} / \partial \overline{Y}_I \\
\mathcal{P}^I_- &= -2 i v T^I u = -i g^{-4}_{_{YM}} \; \partial W / \partial Y_I \,.
\end{aligned} \tag{4.1.61}$$

So the superpotential can be written as:

$$W = i g^4_{_{YM}} Y_I \mathcal{P}^I_- \,. \tag{4.1.62}$$

4.1.4 A particular $\mathcal{N}=4$ theory: $\mathcal{N}=8$

In this section we discuss the further conditions under which the $\mathcal{N}$=4 three dimensional lagrangian previously derived acquires an $\mathcal{N}=8$ supersymmetry. To do that we will compare the four dimensional $\mathcal{N}$=2 lagrangian of [72] with the four dimensional $\mathcal{N}$=4 lagrangian of [75] (rescaled by a factor $\frac{4}{g^2_{YM}}$), whose bosonic part is:

$$\begin{aligned}
\mathcal{L}^{\mathcal{N}=4\ D=4}_{bosonic} &= \frac{1}{g^2_{YM}}\Big\{-F^{\underline{mn}}F_{\underline{mn}}+\frac{1}{4}\nabla^{\underline{m}}\phi^{AB}\nabla_{\underline{m}}\phi^{AB}+\frac{1}{4}\nabla^{\underline{m}}\pi^{AB}\nabla_{\underline{m}}\pi^{AB}+\\
&+\ \frac{1}{64}\left(\left[\phi^{AB},\phi^{CD}\right]\left[\phi^{AB},\phi^{CD}\right]+\left[\pi^{AB},\pi^{CD}\right]\left[\pi^{AB},\pi^{CD}\right]+\right.\\
&+\ \left.2\left[\phi^{AB},\pi^{CD}\right]\left[\phi^{AB},\pi^{CD}\right]\right)\Big\}\ . \qquad (4.1.63)
\end{aligned}$$

The fields π^{AB} and ϕ^{AB} are Lie-algebra valued:

$$\begin{cases} \pi^{AB} &= \pi^{AB}_I t^I \\ \phi^{AB} &= \phi^{AB}_I t^I \end{cases} , \qquad (4.1.64)$$

where t^I are the generators of the gauge group $\mathcal{G}$. They are the real and imaginary parts of the complex field ρ:

$$\begin{cases} \rho^{AB} &= \frac{1}{\sqrt{2}}\left(\pi^{AB}+i\phi^{AB}\right) \\ \overline{\rho}_{AB} &= \frac{1}{\sqrt{2}}\left(\pi^{AB}-i\phi^{AB}\right) \end{cases} . \qquad (4.1.65)$$

ρ^{AB} transforms in the representation **6** of a global $SU(4)$-symmetry of the theory. Moreover, it satisfies the following pseudo-reality condition:

$$\rho^{AB} = -\tfrac{1}{2}i\epsilon^{ABCD}\overline{\rho}_{CD}\,. \qquad (4.1.66)$$

In terms of ρ the lagrangian (4.1.63) can be rewritten as:

$$\mathcal{L}^{\mathcal{N}=8}_{bosonic} = \frac{1}{2g^2_{YM}}\left\{-F^{\underline{mn}}F_{\underline{mn}}+\nabla_{\underline{m}}\overline{\rho}_{AB}\nabla^{\underline{m}}\rho^{AB}+\frac{1}{16}\left[\overline{\rho}_{AB}\rho^{CD}\right]\left[\rho^{AB},\overline{\rho}_{CD}\right]\right\}. \qquad (4.1.67)$$

The $SU(2)$ global symmetry of the $\mathcal{N}$=2, D=4 theory can be diagonally embedded into the $SU(4)$ of the $\mathcal{N}$=4, D=4 theory:

$$\mathcal{U} = \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} \in SU(2)\subset SU(4)\,. \qquad (4.1.68)$$

By means of this embedding, the **6** of SU(4) decomposes as $\mathbf{6}\longrightarrow\mathbf{4}+\mathbf{1}+\mathbf{1}$. Correspondingly, the pseudo-real field ρ can be splitted into:

$$\begin{aligned}
\rho^{AB} &= \begin{pmatrix} 0 & \sqrt{2}Y & g_{YM}u & ig_{YM}\overline{v} \\ -\sqrt{2}Y & 0 & ig_{YM}v & g_{YM}\overline{u} \\ -g_{YM}u & -ig_{YM}v & 0 & -\sqrt{2}\overline{Y} \\ -ig_{YM}\overline{v} & -g_{YM}\overline{u} & \sqrt{2}\overline{Y} & 0 \end{pmatrix} = \\
&= \begin{pmatrix} i\sqrt{2}\sigma^2\otimes Y & g_{YM}Q \\ -g_{YM}Q^T & -i\sqrt{2}\sigma^2\otimes\overline{Y} \end{pmatrix}, \qquad (4.1.69)
\end{aligned}$$

where Y and Q are Lie-algebra valued. The global $SU(2)$ transformations act as:

$$\rho \longrightarrow \mathcal{U}\rho\mathcal{U}^T = \begin{pmatrix} i\sqrt{2}\sigma^2 \otimes Y & g_{_{YM}} UQU^\dagger \\ & \\ -g_{_{YM}} \left(UQU^\dagger\right)^T & -i\sqrt{2}\sigma^2 \otimes \overline{Y} \end{pmatrix}. \tag{4.1.70}$$

Substituting this expression for ρ into (4.1.67) and dimensionally reducing to three dimensions, we obtain the lagrangian (4.1.51). In other words the $\mathcal{N}$=4, D=3 theory is enhanced to $\mathcal{N}$=8 provided the hypermultiplets are in the adjoint representation of $\mathcal{G}$.

4.2 Geometry of Q^{111}, M^{111} and abelian gauge theories

In this section I perform a geometrical analysis of the Q^{111}, M^{111} manifolds deeper than that given in chapter 3. In several points this is only sketched, without proofs. More details and proofs (and more mathematical rigor) can be found in [16].

4.2.1 Q^{111} and M^{111} as fiber bundles and toric manifolds

Q^{111} and M^{111} as fiber bundles

As a premise, I remind the well known result that the complex projective spaces $\mathbb{P}_1$, $\mathbb{P}_2$ are isomorphic to the following coset manifolds:

$$\begin{aligned} \mathbb{P}_1 &= \frac{SU(2)}{U(1)} \\ \mathbb{P}_2 &= \frac{SU(3)}{SU(2)\times U(1)}. \end{aligned} \tag{4.2.1}$$

Now, let us start our geometric analysis.

The manifold

$$Q^{111} = \frac{SU(2)\times SU(2)\times SU(2)}{U(1)\times U(1)} \tag{4.2.2}$$

is a fiber bundle with base space

$$\frac{SU(2)\times SU(2)\times SU(2)}{U(1)\times U(1)\times U(1)} = \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \tag{4.2.3}$$

and fiber $U(1)$, namely,

$$Q^{111} = E\left(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1, U(1)\right). \tag{4.2.4}$$

The description of the fibration encodes the same information as the numbers $(p,q,r) = (1,1,1)$ in the description of chapter 3, as I will show afterwards with the help of toric geometry.

If we extend the fibration from $U(1)$ to

$$U(1)\times \mathbb{R}^+ = \mathbb{C}^*, \tag{4.2.5}$$

we find the conifold (1.2.12) on Q^{111}, with $\mathbb{R}^+$ parametrized by the coordinate r; so we have

$$\mathcal{C}\left(Q^{111}\right) = E\left(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1, \mathbb{C}^*\right). \tag{4.2.6}$$

The manifold

$$M^{111} = \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1)} \tag{4.2.7}$$

is a fiber bundle with base space

$$\frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times U(1) \times U(1) \times U(1)} = \mathbb{P}_2 \times \mathbb{P}_1 \tag{4.2.8}$$

and fiber $U(1)$, namely,

$$M^{111} = E\left(\mathbb{P}_2 \times \mathbb{P}_1, U(1)\right). \tag{4.2.9}$$

The description of the fibration encodes the same information as the numbers $(p,q,r) = (1,1,1)$ in the description of chapter 3, as I will show afterwards with the help of toric geometry.

If we extend the fibration from $U(1)$ to $U(1) \times \mathbb{R}^+$, we find the conifold (1.2.12) on M^{111}:

$$\mathcal{C}\left(M^{111}\right) = E\left(\mathbb{P}_2 \times \mathbb{P}_1, \mathbb{C}^*\right). \tag{4.2.10}$$

Toric manifolds

I do not review here the theory of toric manifolds (for a complete treatment of toric geometry see [76]), I use only few concepts of that theory which are useful in our derivation. In general, a toric manifold can be seen as a manifold

$$\frac{\mathbb{C}^n/F}{\left(\mathbb{C}^*\right)^k}, \tag{4.2.11}$$

where $F \subset \mathbb{C}^n$ is a null measure set. For simplicity, in the following we will not consider F, even if in a rigorous treatment it should be taken into account.

The toric manifold (4.2.11) can be parametrized by n complex coordinates

$$(X_1, X_2, \ldots, X_n) \tag{4.2.12}$$

on which k equivalence relations are defined, describing the action of $\left(\mathbb{C}^*\right)^k$ on $\mathbb{C}^n$:

$$(X_1, X_2, \ldots, X_n) \sim \left((\lambda_1^{p_1^1}\lambda_2^{p_2^1}\cdots\lambda_k^{p_k^1})X_1,\ (\lambda_1^{p_1^2}\lambda_2^{p_2^2}\cdots\lambda_k^{p_k^2})X_2, \ldots, (\lambda_1^{p_1^n}\lambda_2^{p_2^n}\cdots\lambda_k^{p_k^n})X_n\right)$$
$$(\lambda_1, \ldots, \lambda_k) \in \left(\mathbb{C}^*\right)^k. \tag{4.2.13}$$

The matrix

$$\begin{pmatrix} p_1^1 & p_2^1 & \cdots & p_k^1 \\ p_1^2 & p_2^2 & \cdots & p_k^2 \\ \cdots & \cdots & \cdots & \cdots \\ p_1^n & p_2^n & \ldots & p_k^n \end{pmatrix} \tag{4.2.14}$$

codifies the embedding of $\left(\mathbb{C}^*\right)^k$ in $\mathbb{C}^n$. For example, for $k=1$ the matrix with one column whose all the entries are equal to 1 represents the projective space

$$\mathbb{P}^{n-1} = \frac{\mathbb{C}^n}{\mathbb{C}^*}. \tag{4.2.15}$$

The other toric manifolds can be seen as generalizations of the projective spaces.

$\mathcal{C}\left(Q^{111}\right)$ as a toric manifold

The base space of $\mathcal{C}\left(Q^{111}\right)$ is a toric manifold

$$\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 = \frac{\mathbb{C}^2}{\mathbb{C}^*} \times \frac{\mathbb{C}^2}{\mathbb{C}^*} \times \frac{\mathbb{C}^2}{\mathbb{C}^*} = \frac{\mathbb{C}^6}{\left(\mathbb{C}^*\right)^3}, \tag{4.2.16}$$

which can be described by six homogeneous coordinates

$$\begin{gathered} \left(A_i, B_i, C_i\right) \quad i = 1, 2 \\ \left(A_i, B_i, C_i\right) \sim \left(\lambda_1 A_i, \lambda_2 B_i, \lambda_3 C_i\right), \end{gathered} \tag{4.2.17}$$

where each couple of coordinates describes one of the three $\mathbb{P}_1$ factors. This is a toric manifold described by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.2.18}$$

(to be precise, it has six rows and columns, not three, but they are equal in pairs). $\mathcal{C}\left(Q^{111}\right)$, being a fiber bundle with base space $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and fiber $\mathbb{C}^*$, is a toric manifold with one $\mathbb{C}^*$ less in the denominator:

$$\mathcal{C}\left(Q^{111}\right) = E\left(\frac{\mathbb{C}^6}{\left(\mathbb{C}^*\right)^3}, \mathbb{C}^*\right) = \frac{\mathbb{C}^6}{\left(\mathbb{C}^*\right)^2}. \tag{4.2.19}$$

It is simple to see in this context how the fiber bundle structure can implement the information of the embedding of $H = \left(U\left(1\right)\right)^3$ in $G = \left(SU\left(2\right)\right)^3 \times U\left(1\right)$ yielding Q^{111}, namely, the choice $p = q = r = 1$. To describe the fibration, we have to add a further coordinate on the matrix representing the toric manifold, the coordinate y. On the coordinates $\left(A_i, B_i, C_i, y\right)$ there is an action of a $\left(\mathbb{C}^*\right)^3$ group, whose compact part is the action of the $\left(U\left(1\right)\right)^3$ group in Q^{111}. This latter action is generated by (see chapter 3)

$$\begin{aligned} Z' &= -\frac{\mathrm{i}}{2\sqrt{3}}\left(\sigma_3^{(1)} - \sigma_3^{(3)}\right) \\ Z'' &= -\frac{\mathrm{i}}{2\sqrt{3}}\left(-\sigma_3^{(1)} + \sigma_3^{(2)}\right) \\ Z''' &= -\frac{\mathrm{i}}{2\sqrt{3}}\left(\sigma_3^{(1)} + \sigma_3^{(2)} + \sigma_3^{(3)}\right) - \mathrm{i}\frac{\sqrt{3}}{2}Y. \end{aligned} \tag{4.2.20}$$

Then the toric manifold (4.2.19) is described by

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -3 \end{array}\right). \tag{4.2.21}$$

We can eliminate the coordinate y by fixing $\lambda_3 = -1/3y$, getting a matrix with a row and a column less:

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{4.2.22}$$

This matrix, describing the toric form of $\mathcal{C}\left(Q^{111}\right)$, codifies the choice of the embedding $H \subset G$ for the Q^{111} space.

To retain the $\mathbb{Z}_3$ symmetry which exchange the three $\mathbb{P}_1$ factors, we prefer to maintain the three equivalence relations, making them dependent. So the matrix representing $\mathcal{C}\left(Q^{111}\right)$ becomes

$$\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \tag{4.2.23}$$

which means that the equivalence relations are

$$(A_i, B_i, C_i) \sim \left(\lambda_1 \lambda_3^{-1} A_i, \lambda_1^{-1} \lambda_2 B_i, \lambda_2^{-1} \lambda_3 C_i\right), \quad \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*. \tag{4.2.24}$$

The matrix (4.2.23) has rank 2, and the group in the denominator of the coset (4.2.19) can be written as

$$\left(\mathbb{C}^*\right)^2 = \frac{\left(\mathbb{C}^*\right)^3}{\mathbb{C}^*_{diag}}. \tag{4.2.25}$$

We can fix the $\left(\mathbb{R}^+\right)^2 \subset \left(\mathbb{C}^*\right)^2$ gauge in the manifold (4.2.19) by imposing the further condition on the coordinates [5]

$$\begin{aligned} |A_1|^2 + |A_2|^2 &= |B_1|^2 + |B_2|^2 \\ |B_1|^2 + |B_2|^2 &= |C_1|^2 + |C_2|^2 . \end{aligned} \tag{4.2.27}$$

If we interpret the toric description of the cone as a Kähler quotient, the (4.2.27) equations have the interpretation of D–terms.

Summarizing, the cone on Q^{111} can be described by the six complex coordinates (A_i, B_i, C_i) with the constraints (4.2.27) and the equivalence relations

$$(A_i, B_i, C_i) \sim \left(e^{\mathrm{i}\alpha} e^{-\mathrm{i}\gamma} A_i, e^{-\mathrm{i}\alpha} e^{\mathrm{i}\beta} B_i, e^{-\mathrm{i}\beta} e^{\mathrm{i}\gamma} C_i\right), \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{4.2.28}$$

Under the actions of these three $U(1)$'s the coordinates of Q^{111} have the following charges:

$$\begin{aligned} A_i &: \quad (1, -1, 0) \\ B_i &: \quad (0, 1, -1) \\ C_i &: \quad (-1, 0, 1) . \end{aligned} \tag{4.2.29}$$

I stress that one of these $U(1)$'s is decoupled and has no role in our discussion. The group acting on $\mathbb{C}^6 / \left(\mathbb{R}^+\right)^2$ is

$$U(1)^2 = \frac{U(1)^3}{U(1)_{diag}}. \tag{4.2.30}$$

[5] If furthermore we impose

$$|A_1|^2 + |A_2|^2 = 1 \tag{4.2.26}$$

we fix another $\mathbb{R}^+$ gauge, and we get the Q^{111} space itself, but we will not do that, being mainly interested in the cone that is the space transverse to the stack of $M2$–branes.

$\mathcal{C}\left(M^{111}\right)$ as a toric manifold

The base space of $\mathcal{C}\left(M^{111}\right)$ is a toric manifold

$$\mathbb{P}_2 \times \mathbb{P}_1 = \frac{\mathbb{C}^5}{\left(\mathbb{C}^*\right)^2}, \tag{4.2.31}$$

which can be described by five homogeneous coordinates

$$\begin{aligned} &(U_i, V_A) \quad i = 1,2,3; \quad A = 1,2 \\ &(U_i, V_A) \sim (\lambda_1 U_i, \lambda_2 V_A) . \end{aligned} \tag{4.2.32}$$

where U_i and V_A describe the $\mathbb{P}_2$ and $\mathbb{P}_1$ factors respectively. This is a toric manifold described by the matrix

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \tag{4.2.33}$$

(which actually has five rows and columns). $\mathcal{C}\left(M^{111}\right)$, being a fiber bundle with base space $\mathbb{P}_2 \times \mathbb{P}_1$ and fiber $\mathbb{C}^*$, is a toric manifold with one $\mathbb{C}^*$ less in the denominator:

$$\mathcal{C}\left(M^{111}\right) = E\left(\frac{\mathbb{C}^5}{\left(\mathbb{C}^*\right)^2}, \mathbb{C}^*\right) = \frac{\mathbb{C}^5}{\mathbb{C}^*}. \tag{4.2.34}$$

The matrix describing the toric form of $\mathcal{C}\left(M^{111}\right)$ codifies the choice of the embedding $H \subset G$ for the M^{111} space.

As in the Q^{111} case, we can derive the toric description of this manifold, finding how the fiber bundle structure can implement the information of the embedding of $H = U(1) \times U(1)$ in $G = SU(3) \times SU(2) \times U(1)$ in M^{111}, namely, the choice $p = q = r = 1$. To describe the fibration, we have to add a further coordinate on the matrix representing the toric manifold, the coordinate y. On the coordinates (U_i, V_A, y) there is an action of a $\left(\mathbb{C}^*\right)^2$ group, whose compact part is the action of the $\left(U(1)\right)^2$ group in M^{111}. This latter action is generated by (see chapter 3)

$$\begin{aligned} Z' &= \mathrm{i}\sqrt{3}\lambda_8 + \mathrm{i}\sigma_3 - 4\mathrm{i}Y \\ Z'' &= -\mathrm{i}\frac{\sqrt{3}}{2}\lambda_8 + \frac{3}{2}\mathrm{i}\sigma_3. \end{aligned} \tag{4.2.35}$$

Here we have to keep attention to the normalizations. The explicit forms of these generators are

$$\begin{aligned} Z' &= \mathrm{i}\ \mathrm{diag}\left(1,1,-2,1,-1,-4\right) \\ Z'' &= \frac{\mathrm{i}}{2}\ \mathrm{diag}\left(-1,-1,2,3,-3,0\right). \end{aligned} \tag{4.2.36}$$

Then the equivalence relations of these generators on the coordinates (U_i, V_A, y) are (in toric language)

$$\left(\begin{array}{cc|c} 2 & 1 & -4 \\ -2 & 3 & 0 \end{array} \right). \tag{4.2.37}$$

We can eliminate the coordinate y by fixing $\lambda_1 = -1/4y$, getting a matrix with a row and a column less:

$$\left(\begin{array}{cc} -2 & 3 \end{array} \right). \tag{4.2.38}$$

In analogy with the Q^{111} case (and in order to get reasonable results in the non–abelian extension) we prefer to maintain the two equivalence relations, making them dependent. So the matrix representing $\mathcal{C}\left(M^{111}\right)$ becomes

$$\begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix}, \tag{4.2.39}$$

which means that the equivalence relations are

$$(U_i, V_A) \sim \left((\lambda_1)^2 (\lambda_2)^{-2} U_i, (\lambda_1)^{-3} (\lambda_2)^3 V_A\right) \tag{4.2.40}$$

(that is, defining $\rho = \lambda_1/\lambda_2$, $(U_i, V_A) \sim (\rho^2 U_i, \rho^{-3} V_A)$). The matrix (4.2.39) has rank 1, and the group in the denominator of the coset (4.2.34) is

$$\mathbb{C}^* = \frac{(\mathbb{C}^*)^2}{\mathbb{C}^*_{diag}}. \tag{4.2.41}$$

We can fix the $\mathbb{R}^+ \subset \mathbb{C}^*$ gauge in the manifold (4.2.34) by imposing the further condition on the coordinates [6]

$$|U_1|^2 + |U_2|^2 + |U_3|^2 = |V_1|^2 + |V_2|^2 . \tag{4.2.43}$$

If we interpret the toric description of the cone as a Kähler quotient, the (4.2.43) equation has the interpretation of D–term.

Summarizing, the cone on M^{111} can be described by the five complex coordinates (U_i, V_A) with the constraint (4.2.43) and the equivalence relations

$$(U_i, V_A) \sim \left(e^{2i\alpha} e^{-2i\beta} U_i, e^{-3i\alpha} e^{3i\beta} V_A,\right), \quad \alpha, \beta \in \mathbb{R}. \tag{4.2.44}$$

Under the actions of these two dependent $U(1)$'s the coordinates of M^{111} have the following charges:

$$\begin{array}{lcl} U_i & : & (2, -2) \\ V_A & : & (-3, 3)\,. \end{array} \tag{4.2.45}$$

I stress that one of these $U(1)$'s is decoupled and has no role in our discussion. The group acting on $\mathbb{C}^5/\left(\mathbb{R}^+\right)^2$ is

$$U(1) = \frac{U(1)^2}{U(1)_{diag}}. \tag{4.2.46}$$

4.2.2 The abelian theories

The abelian theory for Q^{111}

Given the toric description of $\mathcal{C}\left(Q^{111}\right)$, the identification of an abelian $\mathcal{N} = 2$ gauge theory whose Higgs branch reproduces the conifold is straightforward. The fields appearing in the toric description should represent the fundamental degrees of freedom of the gauge

[6] If furthermore we impose

$$|V_1|^2 + |V_2|^2 = 1 \tag{4.2.42}$$

we fix another $\mathbb{R}^+$ gauge, and we get the M^{111} space itself.

theory. They have definite transformation properties under the gauge group. Out of them we can also build some gauge invariant combinations, which should represent the operators of the conformal theory and which should be matched with the KK spectrum. Geometrically, this corresponds to describing the cone as an affine submanifold of some $\mathbb{C}^p$. This is a standard procedure, which converts the definition of a toric manifold in terms of D-terms to an equivalent one in terms of binomial equations in $\mathbb{C}^p$. In this case, we have an embedding in $\mathbb{C}^8$. We first construct all the $U(1)$ invariants (in this case there are $8 = 2 \times 2 \times 2$ of them)

$$X^{ijk} = A^i B^j C^k, \qquad\qquad i,j,k = 1,2. \tag{4.2.47}$$

They satisfy a set of binomial equations which cut out the image of our conifold $\mathcal{C}(Q^{111})$ in $\mathbb{C}^8$. These equations are actually the 9 quadrics

$$\begin{aligned} 0 &= \left(\epsilon\sigma^A\right)_{ij} X^{i\ell p}\, X^{jmq}\, \epsilon_{\ell m}\, \epsilon_{pq}\,, \\ 0 &= \left(\epsilon\sigma^A\right)_{\ell m} X^{i\ell p}\, X^{jmq}\, \epsilon_{ij}\, \epsilon_{pq}\,, \\ 0 &= \left(\epsilon\sigma^A\right)_{pq} X^{i\ell p}\, X^{jmq}\, \epsilon_{ij}\, \epsilon_{\ell m}\,. \end{aligned} \tag{4.2.48}$$

Indeed, there is a general method to obtain the embedding equations of the cones over algebraic homogeneous varieties based on representation theory. [7] If we want to summarize this general method (see [16]) in few words, we can say the following. Through eq. (4.2.47) we see that the coordinates X^{ijk} of $\mathbb{C}^8$ are assigned to a certain representation $\mathcal{R}$ of the isometry group $SU(2)^3$. In our case such a representation is $\mathcal{R} = (J^{(1)} = \frac{1}{2}, J^{(2)} = \frac{1}{2}, J^{(3)} = \frac{1}{2})$. The products $X^{i_1 j_1 k_1} X^{i_2 j_2 k_2}$ belong to the symmetric product $Sym^2(\mathcal{R})$, which in general branches into various representations, one of highest weight plus several subleading ones. On the cone, however, only the highest weight representation survives while all the subleading ones vanish. Imposing that such subleading representations are zero corresponds to writing the embedding equations. This has far reaching consequences in the conformal field theory, since provides the definition of the chiral ring. In principle all the representations appearing in the k-th symmetric tensor power of $\mathcal{R}$ could correspond to primary conformal operators. Yet the attention should be restricted to those that do not vanish modulo the equations of the cone, namely modulo the ideal generated by the representations of subleading weights. In other words, only the highest weight representation contained in the $Sym^k(\mathcal{R})$ gives a true chiral operator. This is what matches the Kaluza Klein spectra found through harmonic analysis. Two points should be stressed. In general the number of embedding equations is larger than the codimension of the algebraic locus. For instance $8-4 < 9$, i.e. the cone is not a complete intersection. The 9 equations (4.2.48) define the ideal I of $\mathbb{C}[X] := \mathbb{C}[X^{111}, \ldots, X^{222}]$ cutting the cone $\mathcal{C}(Q^{111})$. The second point to stress is the double interpretation of the embedding equations. The fact that Q^{111} leads to $\mathcal{N}=2$ supersymmetry means that it is Sasakian, i.e. it is a circle bundle over a suitable complex three–fold. If considered in $\mathbb{C}^8$ the ideal I cuts out the conifold $\mathcal{C}(Q^{111})$. Being homogeneous, it can also be regarded as cutting out an algebraic variety in $\mathbb{P}^7$. This is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, namely the base of the $U(1)$ fibre-bundle Q^{111}.

It follows from this discussion that the invariant operators X^{ijk} of eq. (4.2.47) can be naturally associated with the building blocks of the gauge invariant composite operators of

[7] The 9 equations were already given in [12] although their representation theory interpretation was not given there.

our CFT. Holomorphic combinations of the X^{ijk} should span the set of chiral operators of the theory. As stated above, the set of embedding equations (4.2.48) imposes restrictions on the allowed representations of $SU(2)^3$ and hence on the existing operators. If we put the definition of X^{ijk} in terms of the fundamental fields A, B, C into the equations (4.2.48), we see that they are automatically satisfied when the theory is abelian. Since we want eventually to promote A, B, C to non-abelian fields, these equations become non-trivial because the fields do not commute anymore. They essentially assert that the chiral operators we may construct out of the X^{ijk} are totally symmetric in the exchange of the various A, B, C, that is they belong to the highest weight representations we mentioned above.

It is clear that the two different geometric descriptions of the conifold, the first in terms of the variables A, B, C and the second in terms of the X, correspond to the two possible parametrization of the moduli space of vacua of an $\mathcal{N} = 2$ theory, one in terms of *vevs* of the fundamental fields and the second in terms of gauge invariant chiral operators.

We notice that this discussion closely parallels the analogous one in [13], [77]. Q^{111} is indeed a close relative of T^{11}.

The abelian theory for M^{111}

Given the toric description of $\mathcal{C}\,(M^{111})$, we can identify the corresponding abelian $\mathcal{N} = 2$ gauge theory. The fields U, V should represent the fundamental degrees of freedom of the gauge theory. As before, we can find a second representation of our manifold in terms of an embedding in some $\mathbb{C}^p$ with coordinates representing the chiral operators of our CFT. In this case, we have an embedding in $\mathbb{C}^{30}$. We again construct all the $U(1)$ invariants (in this case there are 30 of them) and we find that they are assigned to the $(\mathbf{10}, \mathbf{3})$ of $SU(3) \times SU(2)$. The embedding equations of the conifold into $\mathbb{C}^{30}$ correspond to the statement that in the Clebsch–Gordon expansion of the symmetric product $(\mathbf{10}, \mathbf{3}) \otimes_s (\mathbf{10}, \mathbf{3})$ all representations different from the highest weight one should vanish. This yields 325 equations grouped into 5 irreducible representations (see [16]).

As in the Q^{111} case, the $X^{ij\ell|AB}$ can be associated with the building blocks of the gauge invariant composite operators of our CFT and the ideal generated by the embedding equations (see [16]) imposes many restrictions on the existing conformal operators. Actually, as we try to make clear in the explicit comparison with Kaluza Klein data (see section 4.3.4), the entire spectrum is fully determined by the structure of the ideal above. Indeed, as it should be clear from the previous group theoretical description of the embedding equations, the result of the constraints is to select chiral operators which are totally symmetrized in the $SU(3)$ and $SU(2)$ indices.

4.3 The non-abelian theory and the comparison with KK spectrum

In the previous section, we explicitly constructed an abelian theory whose moduli space of vacua reproduces the cone over the two manifolds Q^{111} and M^{111}. These can be easily promoted to non-abelian ones. Once this is done, we can compare the expected spectrum of short operators in the CFT with the KK spectrum.

4.3.1 The case of Q^{111}

The theory for Q^{111} becomes $SU(N) \times SU(N) \times SU(N)$ with three series of chiral fields in the following representations of the gauge group

$$A_i: \quad (\mathbf{N}, \bar{\mathbf{N}}, \mathbf{1}), \qquad B_l: \quad (\mathbf{1}, \mathbf{N}, \bar{\mathbf{N}}), \qquad C_p: \quad (\bar{\mathbf{N}}, \mathbf{1}, \mathbf{N})\,. \tag{4.3.1}$$

The representations of the fundamental fields have been chosen in such a way that they reduce to the abelian theory discussed in the previous section (eq. (4.2.29)). The field content can be conveniently encoded in a quiver diagram, where nodes represent the gauge groups and links matter fields in the bi-fundamental representation of the groups they are connecting. The quiver diagram for Q^{111} is pictured in figure 4.1. The global symmetry

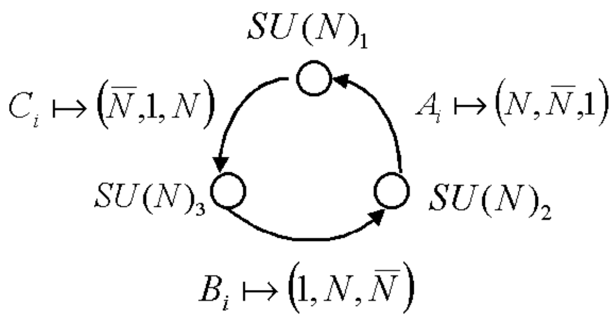


Figure 4.1: Gauge group $SU(N)_1 \times SU(N)_2 \times SU(N)_3$ and colour representation assignments of the fundamental fields A_i, B_j, C_ℓ in the Q^{111} world volume gauge theory.

of the gauge theory is $SU(2)^3$, where each of the doublets of chiral fields transforms in the fundamental representation of one of the $SU(2)$'s.

Notice that we are considering $SU(N)$ gauge group and not the naively expected $U(N)$. The reason is that there is compelling evidence [2], [78], [79] that the $U(1)$ factors are washed out in the near horizon limit. Since in three dimensions $U(1)$ theories may give rise to CFT's in the IR, it is an important point to check whether $U(1)$ factors are described by the AdS-solution or not. A first piece of evidence that the supergravity solutions are dual to $SU(N)$ theories, and not $U(N)$, comes from the absence in the KK spectrum (even in the maximal supersymmetric case) of KK modes corresponding to colour trace of single fundamental fields of the CFT, which are non zero only for $U(N)$ gauge groups. A second evidence is the existence of states dual to baryonic operators in the non-perturbative spectrum of these Type II or M-theory compactifications; baryons exist only for $SU(N)$ groups. We will find baryons in the spectrum of both Q^{111} and M^{111}: this implies that, for the compactifications discussed in this paper, the gauge group of the CFT is $SU(N)$.

In the non-abelian case, we expect that the generic point of the moduli space corresponds to N separated branes. Therefore, the space of vacua of the theory should reduce to the symmetrization of N copies of Q^{111}. To get rid of unwanted light non-abelian degrees of freedom, we would like to introduce, following [13], a superpotential for our theory. Unfortunately, the obvious candidate for this job

$$\epsilon^{ij}\epsilon^{mn}\epsilon^{pq}\mathrm{Tr}(A_i B_m C_p A_j B_n C_q) \tag{4.3.2}$$

is identically zero. Here the close analogy with T^{11} and reference [13] ends.

We consider now the spectrum of KK excitations of Q^{111}. As we have seen in chapter 3, there is a chiral multiplet in the

$$J^{(1)} = \frac{k}{2}\ ; \quad J^{(2)} = \frac{k}{2}\ ; \quad J^{(3)} = \frac{k}{2} \tag{4.3.3}$$

representation of $SU(2)^3$ for each integer value of k, with dimension $E_0 = k$. We naturally associate these multiplets with the series of composite operators

$$\mathrm{Tr}(ABC)^k, \tag{4.3.4}$$

where the $SU(2)$'s indices are totally symmetrized. A first important result, following from the existence of these hypermultiplets in the KK spectrum, is that the dimension of the combination ABC at the superconformal point must be 1.

We see that the predictions from the KK spectrum are in perfect agreement with the geometric discussion in the previous section. Operators which are not totally symmetric in the flavour indices do not appear in the spectrum. The agreement with the proposed CFT, however, is only partial. The chiral operators predicted by supergravity certainly exist in the gauge theory. However, we can construct many more chiral operators which are not symmetric in flavour indices. They do not have any counterpart in the KK spectrum. The superpotential in the case of T^{11} [13] had the double purpose of getting rid of the unwanted non-abelian degrees of freedom and of imposing, via the equations of motion, the total symmetrization for chiral and short operators which is predicted both by geometry and by supergravity. Here, we are not so lucky, since there is no superpotential. We can not consider superpotentials of dimension bigger than that considered before (for example, cubic or quartic in ABC) because the superpotential (4.3.2) is the only one which has dimension compatible with the supergravity predictions. [8] We need to suppose that all the non symmetric operators are not conformal primary. Since the relation between R-charge and dimension is only valid for conformal chiral operators, such operators are not protected and therefore may have enormous anomalous dimension, disappearing from the spectrum. Simple examples of chiral but not conformal operators are those obtained by derivatives of the superpotential. Since we do not have a superpotential here, we have to suppose that both the elimination of the unwanted coloured massless states as well as the disappearing of the non-symmetric chiral operators emerges as a non-perturbative IR effect.

4.3.2 The case of M^{111}

Let us now consider M^{111}. The non-abelian theory is now $SU(N) \times SU(N)$ with chiral matter in the following representations of the gauge group

$$U^i \in Sym^2(\mathbb{C}^N) \otimes Sym^2(\mathbb{C}^{N*}), \qquad\qquad V^A \in Sym^3(\mathbb{C}^{N*}) \otimes Sym^3(\mathbb{C}^N). \tag{4.3.5}$$

The representations of the fundamental fields have been chosen in such a way that they reduce to the abelian theory discussed in the previous section (eq. (4.2.45)), match with the KK spectrum and imply the existence of baryons predicted by supergravity. Comparison with supergravity, which will be made soon, justifies, in particular, the choice of colour symmetric representations.

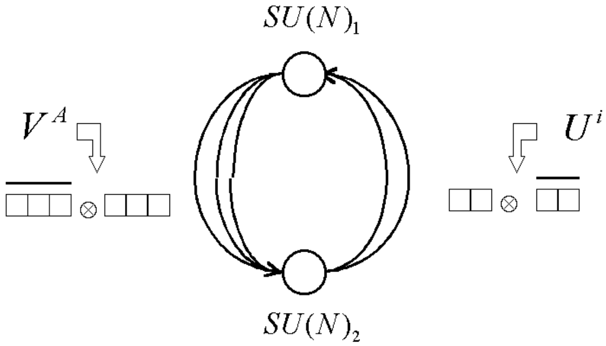


Figure 4.2: Gauge group $U(N)_1 \times U(N)_2$ and colour representation assignments of the fundamental fields V^A and U^i in the M^{111} world volume gauge theory.

The field content can be conveniently encoded in the quiver diagram in figure 4.2.

The global symmetry of the gauge theory is $SU(3) \times SU(2)$, with the chiral fields U and V transforming in the fundamental representation of $SU(3)$ and $SU(2)$, respectively.

We next compare the expectations from gauge theory with the KK spectrum [14]. Let us start with the hypermultiplet spectrum. There is exactly one hypermultiplet in the symmetric representation of $SU(3)$ with $3k$ indices and the symmetric representation of $SU(2)$ with $2k$ indices, namely,

$$[M_1, M_2, 2J] = [3k, 0, 2k] \tag{4.3.6}$$

for each integer $k \geq 1$. The dimension of the operator is $E_0 = 2k$. We naturally identify these states with the totally symmetrized chiral operators

$$\mathrm{Tr}(U^3V^2)^k. \tag{4.3.7}$$

One immediate consequence of the supergravity analysis is that the combination U^3V^2 has dimension 2 at the superconformal fixed point.

Once again, we are not able to write any superpotential of dimension 2. The natural candidate is the dimension two flavour singlet

$$\epsilon_{ijk}\epsilon_{AB}\left(U^iU^jU^kV^AV^B\right)_{\text{colour singlet}} \tag{4.3.8}$$

which however vanishes identically. There is no superpotential that might help in the elimination of unwanted light coloured degrees of freedom and that might eliminate all the non symmetric chiral operators that we can construct out of the fundamental fields. Once again, we have to suppose that, at the superconformal fixed point in the IR, all the non totally symmetric operators are not conformal primaries.

4.3.3 The scalar potential

Let us now consider more closely the scalar potential of the $\mathcal{N} = 2$ world–volume gauge theories we have conjectured to be associated with the Q^{111} and M^{111} compactifications.

[8]For a three dimensional theory to be conformal the dimension of the superpotential must be 2.

In complete generality, the scalar potential of a three dimensional $\mathcal{N}=2$ gauge theory with an arbitrary gauge group and an arbitrary number of chiral multiplets in generic representations of the gauge–group has the form (4.1.50)

$$\begin{aligned}
U(z,\overline{z},M) &= \partial_i W(z)\eta^{ij^*}\partial_{j^*}\overline{W}(\overline{z}) + \\
&+\tfrac{1}{2}g^{IJ}\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right)\left(\overline{z}^{k^*}(T_J)_{k^*l}z^l\right) + \\
&+\overline{z}^{i^*}M^I(T_I)_{i^*j}\eta^{jk^*}M^J(T_J)_{k^*l}z^l + \\
&+2\alpha^2 g_{IJ}M^I M^J + 2\alpha\zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\;I}g_{IJ}M^J + \tfrac{1}{2}\zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\;I}g_{IJ}\zeta^{\widetilde{J}}\mathcal{C}_{\widetilde{J}}^{\;J} + \\
&-2\alpha M^I\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right) - \zeta^{\widetilde{I}}\mathcal{C}_{\widetilde{I}}^{\;I}\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right) .
\end{aligned} \tag{4.3.9}$$

If we put the Chern Simons and the Fayet Iliopoulos terms to zero $\alpha=\zeta^{\widetilde{J}}=0$, the scalar potential becomes the sum of three quadratic forms:

$$U(z,\overline{z},M) = |\partial W(z)|^2 + \tfrac{1}{2}g^{IJ}\,D_I(z,\overline{z}\,)\,D_J(z,\overline{z}) + M^I\,M^J\,K_{IJ}(z,\overline{z}), \tag{4.3.10}$$

where the real functions

$$D^I(z,\overline{z}) = -\overline{z}^{i^*}(T_I)_{i^*j}z^j \tag{4.3.11}$$

are the D–terms, namely the on–shell values of the vector multiplet auxiliary fields, while by definition we have put

$$K_{IJ}(z,\overline{z}) \stackrel{\mathrm{def}}{=} \overline{z}^{i^*}(T_I)_{i^*j}\eta^{jk^*}(T_J)_{k^*l}z^l. \tag{4.3.12}$$

If the quadratic form $M_I\,M_J\,K_{IJ}(z,\bar{z})$ is positive definite, then the vacua of the gauge theory are singled out by the three conditions

$$\begin{aligned}
\frac{\partial W}{\partial z^i} &= 0, && (4.3.13)\\
D^I(z,\bar{z}) &= 0, && (4.3.14)\\
M_I\,M_J\,K_{IJ}(z,\bar{z}) &= 0. && (4.3.15)
\end{aligned}$$

The basic relation between the candidate superconformal gauge theory CFT_3 and the compactifying 7–manifold X_7 that we have used in eq.s (4.2.27, 4.2.43) is that, in the Higgs branch ($\langle M_I\rangle = 0$), the space of vacua of CFT_3, described by eq.s (4.3.13, 4.3.14, 4.3.15), should be equal to the product of N copies of X_7:

$$\text{vacua of gauge theory} = \underbrace{X_7\,\times\,\dots\,\times\,X_7}_{N}/\Sigma_N\,. \tag{4.3.16}$$

Indeed, if there are N M2–branes in the game, each of them can be placed somewhere in X_7 and the vacuum is described by giving all such locations. In order for this to make sense it is necessary that

- The Higgs branch should be distinct from the Coulomb branch
- The vanishing of the D–terms should indeed be a geometric description of (4.3.16).

Let us apply our general formula to the two cases under consideration and see that these conditions are indeed verified.

The scalar potential in the Q^{111} case

Here the gauge group is

$$\mathcal{G} = SU(N)_1 \times SU(N)_2 \times SU(N)_3 \tag{4.3.17}$$

in the non–abelian case $N > 1$ and

$$\mathcal{G} = U(1)_1 \times U(1)_2 \times U(1)_3 \tag{4.3.18}$$

in the abelian case $N = 1$. The chiral fields A_i, B_j, C_ℓ are in the $SU(2)^3$ flavour representations $(\mathbf{2},\mathbf{1},\mathbf{1})$, $(\mathbf{1},\mathbf{2},\mathbf{1})$, $(\mathbf{1},\mathbf{1},\mathbf{2})$ and in the colour $SU(N)^3$ representations $(\mathbf{N},\bar{\mathbf{N}},\mathbf{1})$, $(\mathbf{1},\mathbf{N},\bar{\mathbf{N}})$, $(\bar{\mathbf{N}},\mathbf{1},\mathbf{N})$, respectively (see fig.4.1). We can arrange the chiral fields into a column vector:

$$\vec{z} = \begin{pmatrix} A_i \\ B_j \\ C_\ell \end{pmatrix}. \tag{4.3.19}$$

Naming $(t_I)^\Lambda{}_\Sigma$ the $N \times N$ hermitian matrices such that $\mathrm{i}\, t_I$ span the $SU(N)$ Lie algebra $(I = 1, \ldots, N^2 - 1)$, the generators of the gauge group acting on the chiral fields can be written as follows:

$$\begin{aligned} T_I^{[1]} &= \begin{pmatrix} t_I \otimes \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\mathbf{1} \otimes t_I \end{pmatrix}, \quad T_I^{[2]} = \begin{pmatrix} -\mathbf{1} \otimes t_I & 0 & 0 \\ 0 & t_I \otimes \mathbf{1} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T_I^{[3]} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mathbf{1} \otimes t_I & 0 \\ 0 & 0 & t_I \otimes \mathbf{1} \end{pmatrix}. \end{aligned} \tag{4.3.20}$$

Then the D^2–terms appearing in the scalar potential take the following form:

$$\begin{aligned} D^2\text{-terms} &= \tfrac{1}{2} \Big[\sum_{I=1}^{N^2-1} \big(\bar{A}^i \, (t_I \otimes \mathbf{1}) \, A_i - \bar{C}^i \, (\mathbf{1} \otimes t_I) \, C_i \big)^2 + \\ &+ \sum_{I=1}^{N^2-1} \big(\bar{B}^i \, (t_I \otimes \mathbf{1}) \, B_i - \bar{A}^i \, (\mathbf{1} \otimes t_I) \, A_i \big)^2 + \\ &+ \sum_{I=1}^{N^2-1} \big(\bar{C}^i \, (t_I \otimes \mathbf{1}) \, C_i - \bar{B}^i \, (\mathbf{1} \otimes t_I) \, B_i \big)^2 \Big]. \end{aligned} \tag{4.3.21}$$

The part of the scalar potential involving the gauge multiplet scalars is instead given by:

$$\begin{aligned} M^2\text{–terms} &= M_1^I \, M_1^J \, \big(\bar{A}^i \, (t_I t_J \otimes \mathbf{1}) \, A_i + \bar{C}^i \, (\mathbf{1} \otimes t_I t_J) \, C_i \big) + \\ &+ M_2^I \, M_2^J \, \big(\bar{B}^i \, (t_I t_J \otimes \mathbf{1}) \, B_i + \bar{A}^i \, (\mathbf{1} \otimes t_I t_J) \, A_i \big) + \\ &+ M_3^I \, M_3^J \, \big(\bar{C}^i \, (t_I t_J \otimes \mathbf{1}) \, C_i + \bar{B}^i \, (\mathbf{1} \otimes t_I t_J) \, B_i \big) + \\ &- 2 \, M_1^I \, M_2^J \, \bar{A}^i \, (t_I \otimes t_J) \, A_i - 2 \, M_2^I \, M_3^J \, \bar{B}^i \, (t_I \otimes t_J) \, B_i + \\ &- 2 \, M_3^I \, M_1^J \, \bar{C}^i \, (t_I \otimes t_J) \, C_i. \end{aligned} \tag{4.3.22}$$

In the abelian case we simply get:

$$
\begin{aligned}
D^2\text{-terms} &= \tfrac{1}{2}\Big[\left(|A_1|^2+|A_2|^2-|C_1|^2-|C_2|^2\right)^2+ \\
&\quad+\left(|B_1|^2+|B_2|^2-|A_1|^2-|A_2|^2\right)^2+ \\
&\quad+\left(|C_1|^2+|C_2|^2-|B_1|^2-|B_1|^2\right)^2\Big],
\end{aligned}
\tag{4.3.23}
$$

$$
\begin{aligned}
M^2\text{-terms} &= \Big[\left(|A_1|^2+|A_2|^2\right)(M_1-M_2)^2+ \\
&\quad+\left(|B_1|^2+|B_2|^2\right)(M_2-M_3)^2+ \\
&\quad+\left(|C_1|^2+|C_2|^2\right)(M_3-M_1)^2\Big].
\end{aligned}
\tag{4.3.24}
$$

Eq.s (4.3.23) and (4.3.24) are what we have used in our toric description of Q^{111} as the manifold of gauge–theory vacua in the Higgs branch. Indeed it is evident from eq. (4.3.24) that if we give non vanishing *vev* to the chiral fields, then we are forced to put $< M_1 >=< M_2 >=< M_3 >= m$. Alternatively, if we give non trivial *vevs* to the vector multiplet scalars M_i, then we are forced to put $< A_i >=< B_j >=< C_\ell >= 0$ which confirms that the Coulomb branch is separated from the Higgs branch.

Finally, from eq.s (4.3.21, 4.3.22) we can retrieve the vacua describing N separated branes. Each chiral field has two colour indices and is actually a matrix. Setting

$$
\begin{aligned}
< {A_{i|\Sigma}}^{\Lambda} > &= \delta^{\Lambda}_{\Sigma}\, a^{\Lambda}_i, \\
< {B_{i|\Sigma}}^{\Lambda} > &= \delta^{\Lambda}_{\Sigma}\, b^{\Lambda}_i, \\
< {C_{i|\Sigma}}^{\Lambda} > &= \delta^{\Lambda}_{\Sigma}\, c^{\Lambda}_i,
\end{aligned}
\tag{4.3.25}
$$

a little work shows that the potential (4.3.21) vanishes if each of the N–triplets $a^{\Lambda}_i, b^{\Lambda}_j, c^{\Lambda}_\ell$ separately satisfies the D–term equations, yielding the toric description of a Q^{111} manifold (4.2.27). Similarly, for each abelian generator belonging to the Cartan subalgebra of $U_i(N)$ and having a non trivial action on $a^{\Lambda}_i, b^{\Lambda}_j, c^{\Lambda}_\ell$ we have $< M^{\Lambda}_1 >=< M^{\Lambda}_2 >=< M^{\Lambda}_3 >= m^{\Lambda}$.

The scalar potential in the M^{111} case

Here the gauge group is

$$
\mathcal{G} = SU(N)_1 \times SU(N)_2 \tag{4.3.26}
$$

in the non–abelian case $N > 1$ and

$$
\mathcal{G} = U(1)_1 \times U(1)_2 \tag{4.3.27}
$$

in the abelian case $N = 1$. The chiral fields U_i, V_A are in the $SU(3) \times SU(2)$ flavour representations $(\mathbf{3},\mathbf{1})$, $(\mathbf{1},\mathbf{2})$ respectively. As for colour, they are in the $SU(N)^2$ representations $Sym^2(\mathbb{C}^N) \otimes Sym^2(\mathbb{C}^{N*})$, $Sym^3(\mathbb{C}^{N*}) \otimes Sym^3(\mathbb{C}^N)$ respectively (see fig. 4.2). As before, we can arrange the chiral fields into a column vector:

$$
\vec{z} = \begin{pmatrix} U_i \\ V_A \end{pmatrix}. \tag{4.3.28}
$$

Naming $(t_I^{[3]})^{\Lambda\Sigma\Gamma}{}_{\Xi\Delta\Theta}$ the hermitian matrices generating $SU(N)$ in the three–times symmetric representation and $(t_I^{[2]})^{\Lambda\Sigma}{}_{\Xi\Delta}$ the same generators in the two–times symmetric representation, the generators of the gauge group acting on the chiral fields can be written as follows:

$$T_I^{[1]} \;=\; \begin{pmatrix} t_I^{[2]} \otimes \mathbf{1} & 0 \\ 0 & -\mathbf{1} \otimes t_I^{[3]} \end{pmatrix}, \quad T_I^{[2]} = \begin{pmatrix} -\mathbf{1} \otimes t_I^{[2]} & 0 \\ 0 & t_I^{[3]} \otimes \mathbf{1} \end{pmatrix}. \qquad (4.3.29)$$

Then the D^2–terms appearing in the scalar potential take the following form:

$$\begin{aligned} D^2\text{-terms} \;&=\; \tfrac{1}{2}\Big[\sum_{I=1}^{N^2-1} \left(\bar{U}^i \left(t_I^{[2]} \otimes \mathbf{1}\right) U_i - \bar{V}^A \left(\mathbf{1} \otimes t_I^{[3]}\right) V_A\right)^2 + \\ &\quad + \sum_{I=1}^{N^2-1} \left(\bar{U}^i \left(\mathbf{1} \otimes t_I^{[2]}\right) U_i - \bar{V}^A \left(t_I^{[3]} \otimes \mathbf{1}\right) V_A\right)^2\Big], \end{aligned} \qquad (4.3.30)$$

while the part of the scalar potential involving the gauge multiplet scalars is given by

$$\begin{aligned} M^2\text{–terms} \;&=\; M_1^I\, M_1^J \left(\bar{U}^i \left(t_I^{[2]} t_J^{[2]} \otimes \mathbf{1}\right) U_i + \bar{V}^A \left(\mathbf{1} \otimes t_I^{[3]} t_J^{[3]}\right) V_A\right) + \\ &\quad + M_2^I\, M_2^J \left(\bar{U}^i \left(\mathbf{1} \otimes t_I^{[2]} t_J^{[2]}\right) U_i + \bar{V}^A \left(t_I^{[3]} t_J^{[3]} \otimes \mathbf{1}\right) V_A\right) + \\ &\quad - 2\, M_1^I\, M_2^J\, \bar{U}^i \left(t_I^{[2]} \otimes t_J^{[2]}\right) U_i - \; 2\, M_2^I\, M_1^J\, \bar{V}^A \left(t_I^{[3]} \otimes t_J^{[3]}\right) V_A. \end{aligned} \qquad (4.3.31)$$

In the abelian case we simply get

$$\begin{aligned} D^2\text{-terms} \;&=\; \tfrac{1}{2}\Big\{\left[2\left(|U_1|^2 + |U_2|^2 + |U_3|^2\right) - 3\left(|V_1|^2 + |V_1|^2\right)\right]^2 + \\ &\quad + \left[2\left(|U_1|^2 + |U_2|^2 + |U_3|^2\right) - 3\left(|V_1|^2 + |V_2|^2\right)\right]^2\Big\}, \end{aligned} \qquad (4.3.32)$$

$$M^2\text{-terms} \;=\; \left[4\left(|U_1|^2 + |U_2|^2 + |U_3|^2\right) + 9\left(|V_1|^2 + |V_2|^2\right)\right] (M_1 - M_2)^2. \qquad (4.3.33)$$

Once again from eq.s (4.3.32) and (4.3.33) we see that the Higgs and Coulomb branches are separated. Furthermore, in eq. (4.3.32) we recognize the toric description of M^{111} as the manifold of gauge–theory vacua in the Higgs branch (see eq. (4.3.30)).

As before, from eq.s (4.3.32, 4.3.33) we can retrieve the vacua describing N separated branes. In this case the colour index structure is more involved and we must set

$$\begin{aligned} < U_{i|\Lambda\Lambda}{}^{\Lambda\Lambda} > \;&=\; u_i^{\Lambda}, \\ < V_{A|\Lambda\Lambda\Lambda}{}^{\Lambda\Lambda\Lambda} > \;&=\; v_A^{\Lambda}. \end{aligned} \qquad (4.3.34)$$

A little work shows that the potential (4.3.21) vanishes if each of the N–doublets $u_i^{\Lambda}, v_A^{\Lambda}$ separately satisfies the D–term equations yielding the toric description of a M^{111} manifold (4.3.30). Similarly, for each abelian generator belonging to the Cartan subalgebra of $U_i(N)$ and having a non trivial action on $u_i^{\Lambda}, v_A^{\Lambda}$ we have $< M_1^{\Lambda} >=< M_2^{\Lambda} >= m^{\Lambda}$.

4.3.4 Conformal superfields and comparison with the KK spectrum

Starting from the choice of the fundamental fields of the gauge theory and of the chiral ring (inherited from the geometry of the compact manifold), we can build all sort of candidate conformal superfields for both theories M^{111} and Q^{111}. In the first case, where the full spectrum of $Osp(2|4) \times SU(3) \times SU(2)$ supermultiplets has already been determined through harmonic analysis (see chapter 3, [14]), relying on the conversion vocabulary between AdS_4 bulk supermultiplets and boundary superfields established in section 2.5.3 [15], we can make a detailed comparison of the Kaluza Klein predictions with the candidate conformal superfields available in the gauge theory. In particular we find the gauge theory interpretation of the entire spectrum of short multiplets. The corresponding short superfields are in the right $SU(3) \times SU(2)$ representations and have the right conformal dimensions. Applying the same scheme to the case of Q^{111}, we can use the gauge theory to make predictions about the spectrum of short multiplets one should find in Kaluza Klein harmonic expansions. The partial results already known from harmonic analysis on Q^{111} are in agreement with these predictions.

In addition, looking at the M^{111} spectrum, one finds that there is a rich collection of long multiplets whose conformal dimensions are rational and seem to be protected from acquiring quantum corrections. This is in full analogy with results obtained in the four–dimensional theory associated with the T^{11} manifold [80], [26]. Actually, we find an even larger class of such *rational* long multiplets. For a subclass of them the gauge theory interpretation is clear while for others it is not immediate. Their presence, which seems universal in all coset models, indicates some general protection mechanism that has still to be clarified.

The fundamental superfields of the M^{111} theory are the following ones:

$$\begin{aligned} U^{i|\Lambda\Sigma}{}_{\underline{\Gamma\Delta}}(x,\theta) &= u^{i|\Lambda\Sigma}{}_{\underline{\Gamma\Delta}}(x) + (\lambda^{\alpha}_{u})^{i|\Lambda\Sigma}{}_{\underline{\Gamma\Delta}}(x)\ \theta^{+}_{\alpha}, \\ V^{A|\underline{\Gamma\Delta\Theta}}{}_{\Lambda\Sigma\Pi}(x,\theta) &= v^{A|\underline{\Gamma\Delta\Theta}}{}_{\Lambda\Sigma\Pi}(x) + (\lambda^{\alpha}_{v})^{A|\underline{\Gamma\Delta\Theta}}{}_{\Lambda\Sigma\Pi}(x)\ \theta^{+}_{\alpha}, \end{aligned} \tag{4.3.35}$$

where (i, A) are $SU(3) \times SU(2)$ *flavour* indices, $(\Lambda, \underline{\Lambda})$ are $SU(N) \times SU(N)$ *colour* indices while α is a world volume spinorial index of $SO(1,2)$. The fundamental superfields are chiral superfields, so they satisfy $E_0 = |y_0|$.

U^i is in the fundamental representation $\mathbf{3}$ of $SU(3)_{\text{flavour}}$ and in the $(\square\square, \square\square^{\star})$ of $(SU(N) \times SU(N))_{\text{colour}}$. V^A is in the fundamental representation $\mathbf{2}$ of $SU(2)_{\text{flavour}}$ and in the $(\square\square\square^{\star}, \square\square\square)$ of $(SU(N) \times SU(N))_{\text{colour}}$. In eq.s (4.3.35) we have followed the conventions that lower $SU(N)$ indices transform in the fundamental representation, while upper $SU(N)$ indices transform in the complex conjugate of the fundamental representation.

In the next section, studying the non perturbative baryon state, we will unambiguously establish the conformal weights of the fundamental superfields U, V (or, more precisely, the conformal weights of the Clifford vacua u, v) that are:

$$E_0(u) = y_0(u) = \frac{4}{9}, \quad E_0(v) = y_0(v) = \frac{1}{3}. \tag{4.3.36}$$

For the Q^{111} theory the fundamental superfields are instead the following ones:

$$A_{i_1|\Lambda_1}{}^{\Gamma_2}(x,\theta) = a_{i_1|\Lambda_1}{}^{\Gamma_2}(x) + (\lambda^{\alpha}_{a})_{i_1|\Lambda_1}{}^{\Gamma_2}(x)\ \theta^{+}_{\alpha},$$

$$B_{i_2|\Lambda_2}^{\ \ \Gamma_3}(x,\theta) \;=\; b_{i_2|\Lambda_2}^{\ \ \Gamma_3}(x) + (\lambda_b^\alpha)_{i_2|\Lambda_2}^{\ \ \Gamma_3}(x)\ \theta_\alpha^+,$$
$$C_{i_3|\Lambda_3}^{\ \ \Gamma_1}(x,\theta) \;=\; c_{i_3|\Lambda_3}^{\ \ \Gamma_1}(x) + (\lambda_c^\alpha)_{i_3|\Lambda_3}^{\ \ \Gamma_1}(x)\ \theta_\alpha^+, \tag{4.3.37}$$

where i_ℓ ($\ell = 1,2,3$) are flavour indices of $SU(2)_1 \times SU(2)_2 \times SU(2)_3$, while Λ_ℓ ($\ell = 1,2,3$) are colour indices of $SU(N)_1 \times SU(N)_2 \times SU(N)_3$. Also in this case we know (see next section) their conformal dimensions through the calculation of the conformal dimension of the baryon operators. We have:

$$E_0(a) = E_0(b) = E_0(c) = y_0(a) = y_0(b) = y_0(c) = \frac{1}{3}. \tag{4.3.38}$$

Chiral operators

When the gauge group is $U(1)^N$, there is a simple interpretation for the ring of the chiral superfields: they describe the oscillations of the $M2-$branes in the 7 compact transverse directions, so they should have the form of a parametric description of the manifold. As we have seen, M^{111} embedded in $\mathbb{P}^{29}$, can be parametrized by

$$X^{ijl|AB} = U^i U^j U^k V^A V^B. \tag{4.3.39}$$

Furthermore, the embedding equations can be reformulated in the following way. In a product

$$X^{i_1 j_1 l_1|A_1 B_1}\ X^{i_2 j_2 l_2|A_2 B_2} \dots X^{i_k j_k l_k|A_k B_k} \tag{4.3.40}$$

only the highest weight representation of $SU(3)\times SU(2)$, that is the completely symmetric in the $SU(3)$ indices and completely symmetric in the $SU(2)$ indices, survives. So the ring of the chiral superfields should be composed by superfields of the form

$$\Phi^{(i_1 j_1 l_1 \dots i_k j_k l_k)(A_1 B_1 \dots A_k B_k)} = \underbrace{U^{i_1} U^{j_1} U^{l_1} V^{A_1} V^{B_1} \dots U^{i_k} U^{j_k} U^{l_k} V^{A_k} V^{B_k}}_{k}. \tag{4.3.41}$$

First of all, we note that a product of chiral superfields is always a chiral superfield, that is, a field satisfying the equation (see section 2.5.2)

$$\mathcal{D}_\alpha^+ \Phi = 0, \tag{4.3.42}$$

whose general solution has the form

$$\Phi(x,\theta) = S(x) + \lambda^\alpha(x)\theta_\alpha^+ + \pi(x)\theta^{+\alpha}\theta_\alpha^+. \tag{4.3.43}$$

Following the notation of section 3.4.1, we identify the flavour representations with three nonnegative integers $M_1,\ M_2,\ 2J$. The superfields (4.3.41) are in the same $Osp(2|4) \times SU(3)\times SU(2)$ representations as the bulk hypermultiplets that were determined through harmonic analysis:

$$\begin{cases} M_1 = 3k \\ M_2 = 0 \\ J = k \\ E_0 = y_0 = 2k \end{cases} \qquad k > 0\,. \tag{4.3.44}$$

In particular, it is worth noticing that every block $UUUVV$ is in the $(\square\square\square, \square\square)_{\text{flavour}}$ and has conformal weight

$$3\cdot\left(\frac{4}{9}\right) + 2\cdot\left(\frac{1}{3}\right) = 2, \tag{4.3.45}$$

as in the Kaluza Klein spectrum. As a matter of fact, the conformal weight of a product of chiral fields equals the sum of the weights of the single components, as in a free field theory. This is due to the relation $E_0 = |y_0|$ satisfied by the chiral superfields and to the additivity of the hypercharge.

When the gauge group is promoted to $SU(N)\times SU(N)$, the coordinates become tensors (see (4.3.35)). Our conclusion about the composite operators is that the only primary chiral superfields are those which preserve the structure (4.3.41). So, for example, the lowest lying operator is:

$$U^{\Lambda\Sigma}{}_{i|(\underline{\Lambda\Sigma}}U^{\Gamma\Delta}{}_{j|\underline{\Gamma\Delta}}U^{\Theta\Xi}{}_{\ell|\underline{\Theta\Xi})}V^{\underline{\Lambda\Sigma\Gamma}}{}_{A|(\Lambda\Sigma\Gamma}V^{\underline{\Delta\Theta\Xi}}{}_{B|\Delta\Theta\Xi)}, \tag{4.3.46}$$

where the colour indices of every $SU(N)$ are symmetrized. The generic primary chiral superfield has the form (4.3.41), with all the colour indices symmetrized before being contracted. The choice of symmetrizing the colour indices is not arbitrary: if we impose symmetrization on the flavour indices, it necessarily follows that also the colour indices are symmetrized (see [16] for a proof of this fact). Clearly, the $Osp(2|4)\times SU(3)\times SU(2)$ representations (4.3.44) of these fields are the same as in the abelian case, namely those predicted by the AdS/CFT correspondence.

It should be noted that in the 4–dimensional analogue of these theories, namely in the T^{11} case [13] [26], the restriction of the primary conformal fields to the geometrical chiral ring occurs through the derivatives of the quartic superpotential. As we already noted, in the $D=3$ theories there is no superpotential of dimension 2 which can be introduced and, accordingly, the embedding equations defining the vanishing ideal cannot be given as derivatives of a single holomorphic "function". It follows that there is some other non perturbative and so far unclarified mechanism that suppresses the chiral superfields not belonging to the highest weight representations.

Let us know consider the case of the Q^{111} theory. Here, as already pointed out, the complete Kaluza Klein spectrum is still under construction [18]. Yet the information available in the literature, given at the end of chapter 3, is sufficient to make a comparison between the Kaluza Klein predictions and the gauge theory at the level of the chiral multiplets (and also of the graviton multiplets as I show below). Looking at table 3.13, we learn that in the $AdS_4\times M^{111}$ compactification, each hypermultiplet contains a scalar state S of energy label $E_0 = |y_0|$, which is actually the Clifford vacuum of the representation and corresponds to the world volume field S of eq.(4.3.43). It is reasonable to guess that the same happens in the $AdS_4 \times Q^{111}$ compactification. From the general bosonic mass–formulae (3.1.70), we know that S is related to traceless deformations of the internal metric and its mass is determined by the spectrum of the scalar laplacian on X_7. In (3.1.70) we have

$$m^2_S = H_0 + 176 - 24\,\sqrt{H_0+36} \tag{4.3.47}$$

which, combined with the general AdS_4 relation between scalar masses and energy labels $16(E_0-2)(E_0-1) = m^2$ (2.2.14), yields the formula

$$E_0 = \tfrac{3}{2} + \tfrac{1}{4}\sqrt{180 + H_0 - 24\sqrt{36+H_0}} \tag{4.3.48}$$

for the conformal weight of candidate hypermultiplets in terms of the scalar laplacian eigenvalues. These are already known for Q^{111} (see chapter 3):

$$H_0 = 32\,\left(J^{(1)}\left(J^{(1)}+1\right) + J^{(2)}\left(J^{(2)}+1\right) + J^{(3)}\left(J^{(3)}+1\right) - \tfrac{1}{4}Y^2\right), \tag{4.3.49}$$

where $(J^{(1)}, J^{(2)}, J^{(3)})$ denotes the $SU(2)^3$ flavour representation and y the R–symmetry $U(1)$ charge. From our knowledge of the geometrical chiral ring of Q^{111} and from our calculation of the conformal weights of the fundamental superfields, on the gauge theory side we expect the following chiral operators:

$$\Phi_{i_1 j_1 \ell_1, \dots i_k j_k \ell_k} = \mathrm{Tr}\ (A_{i_1} B_{j_1} C_{\ell_1}\ \dots\ A_{i_k} B_{j_k} C_{\ell_k}) \tag{4.3.50}$$

in the following $Osp(2|4) \times SU(2) \times SU(2) \times SU(2)$ representation:

$$\begin{aligned} Osp(2|4) &\ : \ \text{hypermultiplet with} \left\{ \begin{array}{lcl} E_0 & = & k \\ y_0 & = & k \end{array} \right. && (4.3.51) \\ SU(2) \times SU(2) \times SU(2) &\ : \ J^{(1)} = J^{(2)} = J^{(3)} = \tfrac{1}{2}\, k && (4.3.52) \\ & k \geq 1. \end{aligned}$$

Inserting the representation (4.3.53) into eq. (4.3.49) we obtain $H_0 = 16k^2 + 48k$ and, using this value in eq. (4.3.48), we retrieve the conformal field theory prediction $E_0 = k$. This shows that the hypermultiplet spectrum found in Kaluza Klein harmonic expansions on Q^{111} agrees with the chiral superfields predicted by the conformal gauge theory.

Conserved currents of the world volume gauge theory

The supergravity mass–spectrum on $AdS_4 \times X_7$, where X_7 is an Einstein space admitting $\mathcal{N} = 2$ Killing spinors, contains a number of *ultrashort* or *massless* $Osp(2|4)$ multiplets that correspond to the unbroken local gauge symmetries of the vacuum. These are:

1. The massless $\mathcal{N} = 2$ graviton multiplet
2. The massless $\mathcal{N} = 2$ vector multiplets of the flavour group G'
3. The massless $\mathcal{N} = 2$ vector multiplets associated with the non–trivial harmonic 2–forms of X_7 (the Betti multiplets).

Each of these massless multiplets must have a suitable gauge theory interpretation. Indeed, also on the gauge theory side, the ultra–short multiplets are associated with the symmetries of the theory (global in this case) and are given by the corresponding conserved Noether currents.

We begin with the stress–energy superfield $T_{\alpha\beta}$ which has a pair of symmetric $SO(1,2)$ spinor indices and satisfies the conservation equation

$$\mathcal{D}^+_\alpha T^{\alpha\beta} = \mathcal{D}^-_\alpha T^{\alpha\beta} = 0. \tag{4.3.53}$$

In components, the θ–expansion of this superfield yields the stress energy tensor $T_{\mu\nu}(x)$, the $\mathcal{N} = 2$ supercurrents $j^{A\alpha}_\mu(x)$ $(A = 1, 2)$ and the $U(1)$ R–symmetry current $J^R_\mu(x)$. Obviously $T^{\alpha\beta}$ is a singlet with respect to the flavour group G' and it has

$$E_0 = 2, \quad y_0 = 0, \quad s_0 = 1. \tag{4.3.54}$$

This corresponds to the massless graviton multiplet of the bulk and explains the first entry in the above enumeration.

To each generator of the flavour symmetry group there corresponds, via Noether theorem, a conserved vector supercurrent. This is a scalar superfield $J^I(x,\theta)$ transforming in the adjoint representation of the flavour group G' and satisfying the conservation equations

$$\mathcal{D}^{+\alpha}\mathcal{D}^{+}_{\alpha}J^I = \mathcal{D}^{-\alpha}\mathcal{D}^{-}_{\alpha}J^I = 0. \tag{4.3.55}$$

These superfields have

$$E_0 = 1, \quad y_0 = 0, \quad s_0 = 0 \tag{4.3.56}$$

and correspond to the $\mathcal{N}=2$ massless vector multiplets of G' that propagate on the bulk. This explains the second item of the above enumeration.

In the specific theories under consideration, we can easily construct the flavour currents in terms of the fundamental superfields:

$$\begin{array}{ll}
M^{111} & \left\{\begin{array}{lcl}
J_{SU(3)|j}{}^{i} & = & U^{i|\Lambda\Sigma}{}_{\underline{\Lambda\Sigma}}\,\bar{U}_{j|\Lambda\Sigma}{}^{\underline{\Lambda\Sigma}} - \frac{1}{3}\delta^i_j\,U^{\ell|\Lambda\Sigma}{}_{\underline{\Lambda\Sigma}}\,\bar{U}_{\ell|\Lambda\Sigma}{}^{\underline{\Lambda\Sigma}} \\
J_{SU(2)|B}{}^{A} & = & V^{A|\underline{\Lambda\Sigma\Gamma}}{}_{\Lambda\Sigma\Gamma}\,\bar{V}_{B|\underline{\Lambda\Sigma\Gamma}}{}^{\Lambda\Sigma\Gamma} - \frac{1}{2}\delta^A_B\,V^{C|\underline{\Lambda\Sigma\Gamma}}{}_{\Lambda\Sigma\Gamma}\,\bar{V}_{C|\underline{\Lambda\Sigma\Gamma}}{}^{\Lambda\Sigma\Gamma}
\end{array}\right. \\
Q^{111} & \left\{\begin{array}{lcl}
J_{SU(2)_1|j_1}{}^{i_1} & = & A^{i_1|\Gamma_1}{}_{\Lambda_2}\,\bar{A}_{j_1|\Gamma_1}{}^{\Lambda_2} - \frac{1}{2}\,\delta^{i_1}_{j_1}\,A^{\ell_1|\Gamma_1}{}_{\Lambda_2}\,\bar{A}_{\ell_1|\Gamma_1}{}^{\Lambda_2} \\
J_{SU(2)_2|j_2}{}^{i_2} & = & B^{i_2|\Gamma_2}{}_{\Lambda_3}\,\bar{B}_{j_2|\Gamma_2}{}^{\Lambda_3} - \frac{1}{2}\,\delta^{i_2}_{j_2}\,B^{\ell_2|\Gamma_2}{}_{\Lambda_3}\,\bar{B}_{\ell_2|\Gamma_2}{}^{\Lambda_3} \\
J_{SU(2)_3|j_3}{}^{i_3} & = & C^{i_3|\Gamma_3}{}_{\Lambda_1}\,\bar{C}_{j_3|\Gamma_3}{}^{\Lambda_1} - \frac{1}{2}\,\delta^{i_3}_{j_3}\,C^{\ell_3|\Gamma_3}{}_{\Lambda_1}\,\bar{C}_{\ell_3|\Gamma_3}{}^{\Lambda_1}\,.
\end{array}\right.
\end{array} \tag{4.3.57}$$

These currents satisfy eq.(4.3.55) and are in the right representations of $SU(3)\times SU(2)$. Their hypercharge is $y_0 = 0$. The conformal weight is not the one obtained by a naive sum, being the theory interacting. As we have seen in chapter 2, the conserved currents satisfy $E_0 = |y_0| + 1$, hence $E_0 = 1$.

Let us finally identify the gauge theory superfields associated with the Betti multiplets. As we have stressed, the non abelian gauge theory has $SU(N)^p$ rather than $U(N)^p$ as gauge group. The abelian gauge symmetries that were used to obtain the toric description of the manifold M^{111} and Q^{111} in the one–brane case $N=1$ are not promoted to gauge symmetries in the many brane regime $N\to\infty$. Yet, they survive as exact global symmetries of the gauge theory. The associated conserved currents provide the superfields corresponding to the massless Betti multiplets found in the Kaluza Klein spectrum of the bulk. As the reader can notice, the b_2 Betti number of each manifold always agrees with the number of independent $U(1)$ groups needed to give a toric description of the same manifold. It is therefore fairly easy to identify the Betti currents of our gauge theories. For instance for the M^{111} case the Betti current is

$$J_{\mathrm{Betti}} = 2\,U^{\ell|\Lambda\Sigma}{}_{\underline{\Lambda\Sigma}}\,\bar{U}_{\ell|\Lambda\Sigma}{}^{\underline{\Lambda\Sigma}} - 3\,V^{C|\underline{\Lambda\Sigma\Gamma}}{}_{\Lambda\Sigma\Gamma}\,\bar{V}_{C|\underline{\Lambda\Sigma\Gamma}}{}^{\Lambda\Sigma\Gamma}\,. \tag{4.3.58}$$

The two Betti currents of Q^{111} are similarly written down from the toric description. Since the Betti currents are conserved, according to what shown in chapter 2 they satisfy $E_0 = |y_0| + 1$. Since the hypercharge is zero, we have $E_0 = 1$ and the Betti currents provide the gauge theory interpretation of the massless Betti multiplets.

Gauge theory interpretation of the short multiplets

Using the massless currents above reviewed and the chiral superfields, one has all the building blocks necessary to construct the constrained superfields that correspond to all the short multiplets found in the Kaluza Klein spectrum.

As shown in chapter 2, short $Osp(2|4)$ multiplets correspond to shortened superfields defined imposing a suitable differential constraint, invariant with respect to Poincaré supersymmetry [15]. Using chiral superfields and conserved currents as building blocks, we can construct candidate short superfields that satisfy the appropriate differential constraint and the unitarity bounds (2.4.65). Then we can compare their flavour representations with those of the short multiplets obtained in Kaluza Klein expansions. In the case of the M^{111} theory, where the Kaluza Klein spectrum is known, we find complete agreement and hence we explicitly verify the AdS/CFT correspondence. For the Q^{111} manifold we make instead a prediction in the reverse direction: the gauge theory realization predicts the outcome of harmonic analysis. While we wait for the construction of the complete spectrum [18], we can partially verify the correspondence using the information available at the moment, namely the spectrum of the scalar laplacian.

Superfields corresponding to the short graviton multiplets

The gauge theory interpretation of these multiplets is quite simple. Consider the superfield

$$\Phi_{\alpha\beta}(x,\theta) = T_{\alpha\beta}(x,\theta)\,\Phi_{\text{chiral}}(x,\theta), \tag{4.3.59}$$

where $T_{\alpha\beta}$ is the stress energy tensor (4.3.53) and $\Phi_{\text{chiral}}(x,\theta)$ is a chiral superfield. By construction, the superfield (4.3.59), at least in the abelian case, satisfies the equation

$$\mathcal{D}^{+}_{\alpha}\Phi^{\alpha\beta} = 0 \tag{4.3.60}$$

and then, as shown in chapter 2, it corresponds to a short graviton multiplet on the bulk. It is natural to extend this identification to the non-abelian case.

Given the chiral multiplet spectrum (4.3.44) and the dimension of the stress energy current (4.3.54), we immediately get the spectrum of superfields (4.3.59) for the case M^{111}:

$$\begin{cases} M_1 = 3k \\ M_2 = 0 \\ J = k \\ E_0 = 2k+2, \quad y_0 = 2k \end{cases} \qquad k > 0\,. \tag{4.3.61}$$

This exactly coincides with the spectrum of short graviton multiplets found in Kaluza Klein theory through harmonic analysis.

For the Q^{111} case the same analysis gives the following prediction for the short graviton multiplets:

$$\begin{cases} J^{(1)} = J^{(2)} = J^{(3)} = \frac{1}{2}k \\ E_0 = k+2, \quad y_0 = k \end{cases} \quad k > 0\,. \tag{4.3.62}$$

We can make a consistency check on this prediction just relying on the spectrum of the laplacian (4.3.49). Indeed, looking at table 3.10, we see that in a short graviton multiplet the mass of the spin two particle is

$$m_h^2 = 16y_0(y_0+3). \tag{4.3.63}$$

Looking instead at equation (3.1.70), we see that such a mass is equal to the eigenvalue of the scalar laplacian $m_h^2 = H_0$. Therefore, for consistency of the prediction (4.3.62), we should have $H_0 = 16k(k+3)$ for the representation $J^{(1)} = J^{(2)} = J^{(3)} = k/2; Y = k$. This is indeed the value provided by eq. (4.3.49).

It should be noted that when we write the operator (4.3.59), it is understood that *all colour indices are symmetrized before taking the contraction.*

Superfields corresponding to the short vector multiplets

Consider next the superfields of the following type:

$$\Phi(x,\theta) = J(x,\theta)\,\Phi_{\text{chiral}}(x,\theta), \tag{4.3.64}$$

where J is a conserved vector current of the type analyzed in eq. (4.3.57) and Φ_{chiral} is a chiral superfield. By construction, the superfield (4.3.64), at least in the abelian case, satisfies the constraint

$$\mathcal{D}^{+\alpha}\mathcal{D}^{+}_{\alpha}\Phi = 0 \tag{4.3.65}$$

and then, according to the analysis of section 2.5.3, it can describe a short vector multiplet propagating into the bulk.

In principle, the flavour irreducible representations occurring in the superfield (4.3.64) are those originating from the tensor product decomposition

$$ad \otimes \mathcal{R}_{\rho_k} = \mathcal{R}_{\chi_{max}} \oplus \sum_{\chi<\chi_{max}} \mathcal{R}_{\chi}, \tag{4.3.66}$$

where ad is the adjoint representation, ρ_k is the flavour weight of the chiral field at level k, χ_{max} is the highest weight occurring in the product $ad \otimes \mathcal{R}_{\rho_k}$ and $\chi < \chi_{max}$ are the lower weights occurring in the same decomposition.

Let us assume that the quantum mechanism that suppresses all the candidate chiral superfields of subleading weight does the same suppression also on the short vector superfields (4.3.64). Then in the sum appearing on the l.h.s of eq. (4.3.66) we keep only the first term and, as we show in a moment, we reproduce the Kaluza Klein spectrum of short vector multiplets. As we see, there is just a universal rule that presides at the selection of the flavour representations in all sectors of the spectrum. It is the restriction to the maximal weight. This is the group theoretical implementation of the ideal that defines the conifold as an algebraic locus in $\mathbb{C}^p$. We already pointed out that, differently from the $D = 4$ analogue of these conformal gauge theories, the ideal cannot be implemented through a superpotential. An equivalent way of imposing the result is to assume that the colour indices have to be completely symmetrized: such a symmetrization automatically selects the highest weight flavour representations.

Let us now explicitly verify the matching with Kaluza Klein spectra. We begin with the M^{111} case. Here the highest weight representations occurring in the tensor product of the adjoint $(M_1 = M_2 = 1, J = 0) \oplus (M_1 = M_2 = 0, J = 1)$ with the chiral spectrum (4.3.44) are $M_1 = 3k+1, M_2 = 1, J = k$ and $M_1 = k, M_2 = 0, J = k+1$. Hence the spectrum of vector fields (4.3.64) limited to highest weights is given by the following list of $Osp(2|4) \times SU(2) \times SU(3)$ UIRs:

$$\begin{cases} M_1 = 3k+1 \\ M_2 = 1 \\ J = k \\ E_0 = 2k+1, \quad y_0 = 2k \end{cases} \qquad k > 0 \tag{4.3.67}$$

and

$$\begin{cases} M_1 = 3k \\ M_2 = 0 \\ J = k+1 \\ E_0 = 2k+1, \quad y_0 = 2k \end{cases} \qquad k > 0\,. \tag{4.3.68}$$

This is precisely the result found in chapter 3.

For the Q^{111} case our gauge theory realization predicts the following short vector multiplets:

$$\begin{cases} J^{(1)} = \frac{1}{2}k + 1 \\ J^{(2)} = \frac{1}{2}k \\ J^{(3)} = \frac{1}{2}k \\ E_0 = k+1, \quad y_0 = k \end{cases} \qquad k > 0 \tag{4.3.69}$$

and all the other are obtained from (4.3.69) by permuting the role of the three $SU(2)$ groups. Looking at table 3.13, we see that in the $\mathcal{N}=2$ short multiplet emerging from M–theory compactification on $AdS_4 \times M^{111}$ the lowest energy state is a scalar S, and we guess that the same happens in the $X_7 = Q^{111}$ case. It has squared mass

$$m_S^2 = 16 y_0 (y_0 - 1). \tag{4.3.70}$$

Hence, recalling eq. (4.3.47) and combining it with (4.3.70), we see that for consistency of our predictions we must have

$$H_0 + 176 - 24\sqrt{H_0 + 36} = 16k(k-1) \tag{4.3.71}$$

for the representations (4.3.69). The quadratic equation (4.3.71) implies $H_0 = 16k^2 + 80k + 64$ which is precisely the result obtained by inserting the values (4.3.62) into Pope's formula (4.3.49) for the laplacian eigenvalues. Hence, also the short vector multiplets seems to follow a general pattern identical in all $\mathcal{N}=2$ compactifications.

We can finally wonder why there are no short vector multiplets obtained by multiplying the Betti currents with chiral superfields. The answer might be the following. From the flavour view point these would not be highest weight representations occurring in the tensor product of the constituent fundamental superfields. Hence they are suppressed from the spectrum.

Superfields corresponding to the short gravitino multiplets

The spectrum of M^{111} derived in chapter 3 contains various series of short gravitino multiplets. We can provide their gauge theory interpretation through the following superfields. Consider:

$$\begin{aligned} &\Phi'^{(ii_1j_1\ell_1\dots i_kj_k\ell_k)(AC_1D_1\dots C_kD_k)}_{\alpha jB} = \\ &= \left(U\bar{U}\left(\mathcal{D}^+_\alpha V\bar{V}\right) + V\bar{V}\left(\mathcal{D}^+_\alpha U\bar{U}\right)\right)^{i\ \ A}_{\ j\ \ B} \underbrace{U^{i_1}U^{j_1}U^{\ell_1}V^{C_1}V^{D_1}\dots U^{i_k}U^{j_k}U^{\ell_k}V^{C_k}V^{D_k}}_{k} \end{aligned} \tag{4.3.72}$$

and

$$\begin{aligned} &\Phi''^{(ij\ell i_1j_1\ell_1\dots i_kj_k\ell_k)(C_1D_1\dots C_kD_k)}_{\alpha} = \\ &= \left(U^iU^jU^\ell V^A \mathcal{D}^-_\alpha V^B \epsilon_{AB}\right) \underbrace{U^{i_1}U^{j_1}U^{\ell_1}V^{C_1}V^{D_1}\dots U^{i_k}U^{j_k}U^{\ell_k}V^{C_k}V^{D_k}}_{k}, \end{aligned} \tag{4.3.73}$$

where all the colour indices are symmetrized before being contracted. By construction the superfields (4.3.72,4.3.73), at least in the abelian case, satisfy the equation

$$\mathcal{D}^+_\alpha \Phi^\alpha = 0 \tag{4.3.74}$$

and then, as explained in section 2.5.2, they correspond to short gravitino multiplets propagating on the bulk. We can immediately check that their highest weight flavour representations yield the spectrum of $Osp(2|4) \times SU(2) \times SU(3)$ short gravitino multiplets. Indeed for (4.3.72),(4.3.73) we respectively have:

$$\begin{cases} M_1 = 3k+1 \\ M_2 = 1 \\ J = k+1 \\ E_0 = 2k + \frac{5}{2}, \quad y_0 = 2k+1 \end{cases} \qquad k \geq 0\,, \tag{4.3.75}$$

and

$$\begin{cases} M_1 = 3k+3 \\ M_2 = 0 \\ J = k \\ E_0 = 2k + \frac{5}{2}, \quad y_0 = 2k+1 \end{cases} \qquad k \geq 0\,. \tag{4.3.76}$$

We postpone the analysis of short gravitino multiplets on Q^{111} to [18] since this requires a more extended knowledge of the spectrum.

Long multiplets with rational protected dimensions

Let us now observe that, in complete analogy to what happens for the T^{11} conformal spectrum one dimension above [80], [26], also in the case of M^{111} there is a large class of long multiplets with rational conformal dimensions. Actually this seems to be a general phenomenon in all Kaluza Klein compactifications on homogeneous spaces G/H. Indeed, although the Q^{111} spectrum is not yet completed [18], we can already see from its laplacian spectrum (4.3.49) that a similar phenomenon occurs also there. More precisely, while the short multiplets saturate the unitarity bound and have a conformal weight related to the hypercharge and maximal spin by equations (2.4.65), the *rational long multiplets* satisfy a quantization condition of the conformal dimension of the following form

$$E_0 = |y_0| + s_0 + 1 + \lambda, \quad \lambda \in \mathbb{N}. \tag{4.3.77}$$

Inspecting the M^{111} spectrum, we find the following long rational multiplets:

- *Long rational graviton multiplets*

  In the series
  $$\begin{cases} M_1 = 0,\ M_2 = 3k,\ J = k+1 \\ M_1 = 1,\ M_2 = 3k+1,\ J = k \end{cases} \tag{4.3.78}$$
  and conjugate ones we have
  $$y_0 = 2k, \quad E_0 = 2k+3 = |y_0| + 3 \tag{4.3.79}$$
  corresponding to
  $$\lambda = 1. \tag{4.3.80}$$

- *Long rational gravitino multiplets*

  In the series of representations

$$M_1 = 1,\; M_2 = 3k + 1,\; J = k + 1 \tag{4.3.81}$$

  (and conjugate ones) for the gravitino multiplets of type χ^- we have

$$y_0 = 2k + 1, \quad E_0 = 2k + \frac{9}{2} = |y_0| + \frac{7}{2}, \tag{4.3.82}$$

  while in the series

$$M_1 = 0,\; M_2 = 3k + 3,\; J = k \tag{4.3.83}$$

  (and conjugate ones) for the same type of gravitinos we get

$$y_0 = 2k + 1, \quad E_0 = 2k + \frac{9}{2} = |y_0| + \frac{7}{2}. \tag{4.3.84}$$

  Both series fit into the quantization rule (4.3.77) with:

$$\lambda = 2. \tag{4.3.85}$$

- *Long rational vector multiplets*

  In the series

$$M_1 = 0,\; M_2 = 3k,\; J = k \tag{4.3.86}$$

  (and conjugate ones) for the vector multiplets of type W we have

$$y_0 = 2k, \quad E_0 = 2k + 4 = |y_0| + 4, \tag{4.3.87}$$

  that fulfills the quantization condition (4.3.77) with

$$\lambda = 3. \tag{4.3.88}$$

  For the same vector multiplets of type W, in the series

$$\begin{cases} M_1 = 0,\; M_2 = 3k,\; J = k + 1 \\ M_1 = 1,\; M_2 = 3k + 1,\; J = k \end{cases} \tag{4.3.89}$$

  (and conjugate ones) we have

$$y_0 = 2k, \quad E_0 = 2k + 10 = |y_0| + 10, \tag{4.3.90}$$

  that satisfies the quantization condition (4.3.77) with

$$\lambda = 9. \tag{4.3.91}$$

The generalized presence of these rational long multiplets hints at various still unexplored quantum mechanisms that, in the conformal field theory, protect certain operators from acquiring anomalous dimensions. At least for the long graviton multiplets, characterized by $\lambda = 1$, the corresponding protected superfields can be guessed, in analogy whith [25].

If we take the superfield of a short vector multiplet $J(x,\theta)\,\Phi_{\text{chiral}}(x,\theta)$ and we multiply it by the stress–energy superfield $T_{\alpha\beta}(x,\theta)$, namely if we consider a superfield of the form

$$\Phi \sim \text{conserved vector current} \times \text{stress energy tensor} \times \text{chiral operator}, \tag{4.3.92}$$

we reproduce the right $Osp(2|4) \times SU(3) \times SU(2)$ representations of the long rational graviton multiplets of M^{111}. The soundness of such an interpretation can be checked by looking at the graviton multiplet spectrum on Q^{111}. This is already available since it is once again determined by the laplacian spectrum. Applying formula eq. (4.3.92) to the Q^{111} gauge theory leads to predict the following spectrum of long rational multiplets:

$$\begin{cases} J^{(1)} = \frac{1}{2}k + 1 \\ J^{(2)} = \frac{1}{2}k \\ J^{(3)} = \frac{1}{2}k \\ E_0 = k+1, \quad y_0 = k \end{cases} \qquad k > 0 \tag{4.3.93}$$

and all the other are obtained from (4.3.93) by permuting the role of the three $SU(2)$ groups. Looking at table 3.7, we see that in a graviton multiplet the spin two particle has mass

$$m_h^2 = 16(E_0+1)(E_0-2), \tag{4.3.94}$$

which for the candidate multiplets (4.3.94) yields

$$m_h^2 = 16(k+4)(k+1). \tag{4.3.95}$$

On the other hand, looking at equation (3.1.70) we see that the squared mass of the graviton is just the eigenvalue of the scalar laplacian $m_h^2 = H_0$. Applying formula (4.3.49) to the representations of (4.3.93) we indeed find

$$H_0 = 16k^2 + 80k + 64 = 16(k+4)(k+1). \tag{4.3.96}$$

It appears, therefore, that the generation of rational long graviton multiplets is based on the universal mechanism codified by the ansatz (4.3.92), proposed in [25] and applicable to all compactifications. Why these superfields have protected conformal dimensions is still to be clarified within the framework of the superconformal gauge theory. The superfields leading to rational long multiplets with much higher values of λ, like the cases $\lambda = 3$ and $\lambda = 9$ that we have found, are more difficult to guess. Yet their appearance seems to be a general phenomenon and this, as we have already stressed, hints at general protection mechanisms that have still to be investigated.

4.4 The baryons

There is one important property that M^{111}, Q^{111} and T^{11} share. These manifolds have non-zero Betti numbers ($b_2 = b_5 = 2$ for Q^{111}, $b_2 = b_5 = 1$ for M^{111} and $b_2 = b_3 = 1$ for T^{11}). This implies the existence of non-perturbative states in the supergravity spectrum associated with branes wrapped on non-trivial cycles. They can be interpreted as baryons in the CFT [79] [81].

The existence of non-zero Betti numbers implies the existence of new global $U(1)$ symmetries which do not come from the geometrical symmetries of the coset manifold, as was

pointed out long time ago. The massless vector multiplets associated with these symmetries were discovered in [22], [8]. They have the property that the entire KK spectrum is neutral and only non-perturbative states can be charged. The massless vectors, dual to the conserved currents, arise from the reduction of the 11-dimensional 3-form along the non-trivial 2-cycles. This definition implies that non-perturbative objects made with M2 and M5 branes are charged under these $U(1)$ symmetries.

We can identify the Betti multiplets with baryonic symmetries. This was first pointed out in [82], [26] for the case of T^{11} and discussed for orbifold models in [65]. The existence of baryons in the proposed CFT's is due to the choice of $SU(N)$ (as opposed to $U(N)$) as gauge group. In the $SU(N)$ case, we can form the gauge invariant operators $\det(A)$, $\det(B)$ and $\det(C)$ for Q^{111} and $\det(U)$ and $\det(V)$ for M^{111} (defined below). The baryon symmetries act on fields in the same way as the $U(1)$ factors that we used for defining our abelian theories in section 4.2.2. They disappeared in the non-abelian theory associated to the conifolds, but the very same fact that they can be consistently incorporated in the theory means that they must exist as global symmetries. It is easy to check that no operator corresponding to KK states is charged under these $U(1)$'s. The reason is that the KK spectrum is made out with the combinations $X = ABC$ or $X = U^3V^2$ defined in section 4.2.2 which, by definition, are $U(1)$ invariant variables. The only objects that are charged under the $U(1)$ symmetries are the baryons.

Baryons have dimensions which diverge with N and can not appear in the KK spectrum. They are indeed non-perturbative objects associated with wrapped branes [79], [81]. We see that the baryonic symmetries have the right properties to be associated with the Betti multiplets: the only charged objects are non-perturbative states. This identification can be strengthened by noticing that the only non-perturbative branes in M-theory have an electric or magnetic coupling to the eleven dimensional three-form. Since for our manifolds, both b_2 and b_5 are greater than 0, we have the choice of wrapping both M2 and M5-branes. M2 branes wrapped around a non-trivial two-cycle are certainly charged under the massless vector in the Betti multiplet which is obtained by reducing the three-form on the same cycle. Since a non-trivial 5-cycle is dual to a 2-cycle, a similar remark applies also for M5-branes. We identify M5-branes as baryons because they have a mass (and therefore a conformal dimension) which, as we will show, goes like N.

What follows from the previous discussion and is probably quite general, is that there is a close relation between the $U(1)$'s entering the brane construction of the gauge theory, the baryonic symmetries and the Betti multiplets. The previous remarks apply as well to CFT associated with orbifolds of $AdS_4 \times S^7$. In the case of T^{11},Q^{111} and M^{111}, the baryonic symmetries are also directly related to the $U(1)$'s entering the toric description of the manifold.

4.4.1 Dimension of the fundamental superfields and the baryon operators

A crucial check of our conjectured conformal gauge theories comes from a direct computation of the conformal weight of the fundamental superfields

$$\text{fundamental superfields} = \begin{cases} U^i \quad V^A & \text{in the } M^{111} \text{ theory} \\ \\ A_i \quad B_j \quad C_\ell & \text{in the } Q^{111} \text{ theory} \end{cases} \tag{4.4.1}$$

whose colour index structure and θ-expansion are explicitly given in the formulae (4.3.35), (4.3.37). If the non–abelian gauge theory has the $SU(N) \times \ldots \times SU(N)$ gauge groups illustrated by the quiver diagrams of fig.s 4.1 and 4.2, then we can consider the following chiral operators:

$$\begin{aligned}
\det U &\equiv U^{\Lambda^2_1 \Sigma^2_1}_{i_1|\Lambda^1_1 \Sigma^1_1} \ldots U^{\Lambda^2_N \Sigma^2_N}_{i_N|\Lambda^1_N \Sigma^1_N} \, \epsilon^{\Lambda^1_1 \ldots \Lambda^1_N} \, \epsilon^{\Sigma^1_1 \ldots \Sigma^1_N} \epsilon_{\Lambda^2_1 \ldots \Lambda^2_N} \, \epsilon_{\Sigma^2_1 \ldots \Sigma^2_N} && (4.4.2) \\
\det V &\equiv V^{\Lambda^2_1 \Sigma^2_1 \Gamma^2_1}_{A_1|\Lambda^1_1 \Sigma^1_1 \Gamma^1_1} \ldots V^{\Lambda^2_N \Sigma^2_N \Gamma^2_N}_{A_N|\Lambda^1_N \Sigma^1_N \Gamma^1_N} \, \epsilon^{\Lambda^1_1 \ldots \Lambda^1_N} \, \epsilon^{\Sigma^1_1 \ldots \Sigma^1_N} \epsilon^{\Gamma^1_1 \ldots \Gamma^1_N} \epsilon_{\Lambda^2_1 \ldots \Lambda^2_N} \, \epsilon_{\Sigma^2_1 \ldots \Sigma^2_N} \epsilon_{\Gamma^2_1 \ldots \Gamma^2_N} && (4.4.3) \\
\det A &\equiv A^{\Lambda^2_1}_{i_1|\Lambda^1_1} \ldots A^{\Lambda^2_N}_{i_N|\Lambda^1_N} \, \epsilon^{\Lambda^1_1 \ldots \Lambda^1_N} \, \epsilon_{\Lambda^2_1 \ldots \Lambda^2_N} && (4.4.4) \\
\det B &\equiv B^{\Lambda^3_1}_{i_1|\Lambda^2_1} \ldots B^{\Lambda^3_N}_{i_N|\Lambda^2_N} \, \epsilon^{\Lambda^2_1 \ldots \Lambda^2_N} \, \epsilon_{\Lambda^3_1 \ldots \Lambda^3_N} && (4.4.5) \\
\det C &\equiv C^{\Lambda^1_1}_{i_1|\Lambda^3_1} \ldots C^{\Lambda^1_N}_{i_N|\Lambda^3_N} \, \epsilon^{\Lambda^3_1 \ldots \Lambda^3_N} \, \epsilon_{\Lambda^1_1 \ldots \Lambda^1_N} \,. && (4.4.6)
\end{aligned}$$

If these operators are truly chiral primary fields, then their conformal dimensions are obviously given by

$$\begin{array}{rclcrclcrcl}
h[\det U] &=& h[U] \times N &;& h[\det V] &=& h[V] \times N \\
h[\det A] &=& h[A] \times N &;& h[\det B] &=& h[B] \times N &;& h[\det C] &=& h[C] \times N
\end{array} \quad (4.4.7)$$

and their flavour representations are:

$$\begin{aligned}
\det U &\Rightarrow (M_1 = N, M_2 = 0, J = 0), && (4.4.8) \\
\det V &\Rightarrow (M_1 = 0, M_2 = 0, J = N/2), && (4.4.9) \\
\det A &\Rightarrow (J^{(1)} = N/2, J^{(2)} = 0, J^{(3)} = 0), && (4.4.10) \\
\det B &\Rightarrow (J^{(1)} = 0, J^{(2)} = N/2, J^{(3)} = 0), && (4.4.11) \\
\det C &\Rightarrow (J^{(1)} = 0, J^{(2)} = 0, J^{(3)} = N/2) \,. && (4.4.12)
\end{aligned}$$

The interesting fact is that the conformal operators (4.4.2,...,4.4.6) can be reinterpreted as solitonic supergravity states obtained by wrapping a 5–brane on a non–trivial supersymmetric 5–cycle. This gives the possibility of calculating directly the mass of such states and, as a byproduct, the conformal dimension of the individual fundamental superfields. All what is involved is a geometrical information, namely the ratio of the volume of the 5–cycles to the volume of the entire compact 7–manifold. In addition, studying the stability subgroup of the supersymmetric 5–cycles, we can also verify that the gauge–theory predictions (4.4.8,...,4.4.12) for the flavour representations are the same one obtains in supergravity looking at the state as a wrapped solitonic 5–brane.

To establish these results we need to derive a general mass–formula for baryonic states corresponding to wrapped 5–branes. This formula is obtained by considering various relative normalizations.

The M2 brane solution and normalizations of the seven manifold metric and volume

Let us write the curvatures of the Freund Rubin solution (1.2.13)

$$\begin{array}{rclcrcl}
R^{mn} &=& -16 e^2 \, E^m \wedge E^n & \Rightarrow & R^{mr}_{nr} &=& -24 \, e^2 \, \delta^m_n \\
\mathcal{R}^{ab} &=& \mathcal{R}^{ab}_{cd} \, \mathcal{B}^c \, \mathcal{B}^d & \text{with} & \mathcal{R}^{ab}_{cb} &=& 12 \, e^2 \, \delta^a_c \\
F^{[4]} &=& e \, \varepsilon_{mnrs} \, E^m \wedge E^n \wedge E^r \wedge E^s \,,
\end{array} \quad (4.4.13)$$

where E^m $(m = 0, 1, 2, 3)$ is the vielbein of anti–de Sitter space AdS_4, R^{mn} is the corresponding curvature 2–form, $\mathcal{B}^a$ $(a = 4, \dots, 10)$ is the vielbein of X_7 and $\mathcal{R}^{ab}$ is the corresponding curvature. In these normalizations, both the internal and space–time vielbeins do not have their physical dimension of a length $[E^m]_{phys} = [\mathcal{B}^a]_{phys} = \ell$, since one has reabsorbed the Planck length l_p into their definition by working in natural units where the $D = 11$ gravitational constant G_{11} has been set equal to $\frac{1}{8\pi}$. Physical units are reinstalled through the following rescaling:

$$\begin{aligned} E^m &= \frac{1}{\kappa^{2/9}} \hat{E}^m, \\ \mathcal{B}^a &= \frac{1}{\kappa^{2/9}} \hat{\mathcal{B}}^a, \\ F^{[4]}_{mnrs} &= \kappa^{11/9} \hat{F}^{[4]}_{mnrs}, \\ \kappa^2 &= 8\pi G_{11} \sim l_p^9. \end{aligned} \tag{4.4.14}$$

After such a rescaling, the relations between the Freund Rubin parameter and the curvature scales for both AdS_4 and X_7 become

$$\text{Ricci}^{AdS}_{\mu\nu} = -2\,\Lambda\, g_{\mu\nu} \tag{4.4.15}$$
$$\text{Ricci}_{\alpha\beta} = \Lambda g_{\alpha\beta} \tag{4.4.16}$$
$$\Lambda \stackrel{\text{def}}{=} 24\, \frac{e^2}{\kappa^{4/9}}. \tag{4.4.17}$$

Note that in eq. (4.4.17) we have used the normalization of the Ricci tensor which is standard in the general relativity literature and is twice the normalization of the Ricci tensor R^{ab}_{cb} appearing in eq. (4.4.13) and in chapter 3. Furthermore eq.s (4.4.13) were written in flat indices while eq.s (4.4.15, 4.4.16) are written in curved indices.

In the solvable coordinates [54], [1] defined in chapters 1, 2, the anti–de Sitter metric is:

$$\begin{aligned} ds^2_{AdS_4} &= R^2_{AdS} \left[\rho^2 \left(-dt^2 + dx_1^2 + dx_2^2 \right) + \frac{d\rho^2}{\rho^2} \right], \\ \text{Ricci}^{AdS}_{\mu\nu} &= -\frac{3}{R^2_{AdS}}\, g_{\mu\nu}, \end{aligned} \tag{4.4.18}$$

which yields the relation anticipated in chapter 1:

$$R_{AdS} = \frac{\kappa^{2/9}}{4\,e} = \frac{1}{2}\sqrt{\frac{6}{\Lambda}}. \tag{4.4.19}$$

As I said, we can consider the exact M2–brane solution of $D = 11$ supergravity that has the cone $\mathcal{C}(X_7)$ over X_7 as transverse space. The $D = 11$ bosonic action can be written as

$$I_{11} = \int d^{11}x \sqrt{-g}\, \left(\frac{R}{\kappa^2} - 3\, \hat{F}^2_{[4]}\right) + 288\sigma \int \hat{F}_{[4]} \wedge \hat{F}_{[4]} \wedge \hat{A}_{[3]} \tag{4.4.20}$$

(where the coupling constant for the last term is $\sigma = \kappa$) and the exact $M2$–brane solution is as follows:

$$ds^2_{M2} = \left(1 + \frac{R^6}{r^6}\right)^{-2/3} (-dt^2 + dx_1^2 + dx_2^2) + \left(1 + \frac{R^6}{r^6}\right)^{1/3} ds^2_{cone},$$

$$
\begin{aligned}
ds^2_{cone} &= dr^2 + r^2\frac{\Lambda}{6}ds^2_{X_7}, \\
A^{[3]} &= dt\wedge dx_1\wedge dx_2\left(1+\frac{R^6}{r^6}\right)^{-1},
\end{aligned}
\tag{4.4.21}
$$

where $ds^2_{X_7}$ is the Einstein metric on X_7, with Ricci tensor as in eq. (4.4.17), and ds^2_{cone} is the corresponding Ricci flat metric on the associated cone. When we go near the horizon, $r\to 0$, the metric (4.4.21) is approximated by

$$
ds^2_{M2}\approx\frac{r^4}{R^4}(-dt^2+dx_1^2+dx_2^2)+R^2\frac{dr^2}{r^2}+R^2\frac{\Lambda}{6}ds^2_{X_7}. \tag{4.4.22}
$$

The Freund Rubin solution $AdS_4\times X_7$ is obtained by setting

$$
\rho=\frac{2}{R^3}r^2 \tag{4.4.23}
$$

and by identifying

$$
R_{AdS}=\frac{R}{2}\quad\Leftrightarrow\quad\Lambda=\frac{6}{R^2}. \tag{4.4.24}
$$

The dimension of the baryon operators

Having fixed the normalizations, we can now compute the mass of a M5-brane wrapped around a non-trivial supersymmetric cycle of X_7 and the conformal dimension of the associated baryon operator.

The parameter R^6 appearing in the M2-solution is obviously proportional to the number N of membranes generating the AdS-background and, by dimensional analysis, to l_p^6. The exact relation for the maximally supersymmetric case $AdS_4\times S^7$ is (see chapter 1)

$$
R_{AdS}=\frac{l_p}{2}\left(2^5\pi^2N\right)^{1/6}\quad\Leftrightarrow\quad R^6=2^5\pi^2Nl_p^6. \tag{4.4.25}
$$

We can easily adapt this formula to the case of $AdS_4\times X_7$ by noticing that, by definition, the number of M2-branes N is determined by the flux of the RR three-form through X_7, $\int_{X_7}*F^{[4]}$. As a consequence, N and the volume of X_7 will appear in all the relevant formulae in the combination $N/\mathrm{Vol}(X_7)$. We therefore obtain the general formula

$$
\sqrt{\frac{\Lambda}{6}}=\frac{1}{R}=\left(\frac{\mathrm{Vol}(X_7)}{\mathrm{Vol}(S^7)}\right)^{1/6}\frac{1}{l_p(2^5\pi^2N)^{1/6}}. \tag{4.4.26}
$$

We can now consider the solitonic particles in AdS_4 obtained by wrapping M2- and M5-branes on the non-trivial 2- and 5-cycles of X_7, respectively. They are associated with boundary operators with conformal dimensions that diverge in the large N limit. The exact dependence on N can be easily estimated. The mass of a p-brane wrapped on a p-cycle is given by $T_p\times\mathrm{Vol}(\mathrm{p-cycle})\sim l_p^{-(p+1)}\Lambda^{-\frac{p}{2}}\sim l_p^{-(p+1)}$. Once the mass of the non-perturbative states is known, the dimension E_0 of the associated boundary operator is given by the relation [9]

$$
m^2=\frac{2\Lambda}{3}(E_0-1)(E_0-2)\simeq\frac{2\Lambda}{3}E_0^2. \tag{4.4.27}
$$

[9]In general $m^2\simeq E_0^2/R^2_{AdS}$; with the conventions of chapter 3, $R_{AdS}=1/4$ and $m^2\simeq E_0^2/16$; whith the convention (4.4.26), $m^2\simeq 2\Lambda/3E_0^2$.

From equation (4.4.26) we learn that $l_p \sim N^{-1/6}$. We see that M2-branes correspond to operators with dimension $\sqrt{N}$ while M5-branes to operators with dimension of order N. The natural candidates for the baryonic operators we are looking for are therefore the wrapped five-branes.

We can easily write a more precise formula for the dimension of the baryonic operator associated with a wrapped M5-brane, following the analogous computation in [81]. For this, we need the exact expression for the M5 tension which can be found, for example, in [83]. We find

$$m = \frac{1}{(2\pi)^5 l_p^6} \mathrm{Vol}(5 - \mathrm{cycle}). \tag{4.4.28}$$

Using equations (4.4.26), (4.4.27), and substituting $V(S^7) = \pi^4 R^7/3$, we obtain the formula for the dimension of a baryon,

$$E_0 = \frac{\pi N}{\Lambda} \frac{\mathrm{Vol}(5 - \mathrm{cycle})}{\mathrm{Vol}(X_7)}, \tag{4.4.29}$$

where the volume is evaluated with the internal metric normalized so that (4.4.16) is true.

As a check, we can compute the dimension of a Pfaffian operator in the $\mathcal{N} = 8$ theory with gauge group $SO(2N)$. The theory contains adjoint scalars which can be represented as antisymmetric matrices ϕ_{ij} and we can form the gauge invariant baryonic operator $\epsilon_{i_1,\ldots,i_{2N}} \phi_{i_1 i_2} \ldots \phi_{i_{2N-1} i_{2N}}$ with dimension $N/2$. The internal manifold is $\mathbb{RP}^7$ [79], [84], a supersymmetric preserving $\mathbb{Z}_2$ projection of original $AdS_4 \times S^7$ case, corresponding to the $SU(N)$ gauge group. We obtain the Pfaffian by wrapping an M5-brane on a $\mathbb{RP}^5$ submanifold. Equation (4.4.29) gives

$$E_0 = \frac{\pi N}{\Lambda} \frac{\mathrm{Vol}\mathbb{RP}^5}{\mathrm{Vol}\mathbb{RP}^7} = \frac{\pi N}{\Lambda} \frac{\mathrm{Vol} S^5}{\mathrm{Vol} S^7} = N/2, \tag{4.4.30}$$

as expected.

4.4.2 The case of M^{111}

Cohomology of M^{111}

Let us now compute the cohomology of M^{111}. The first Chern class of L is $c_1 = 2\omega_1 + 3\omega_2$, where ω_1 (resp. ω_2) is the generator of the second cohomology group of $\mathbb{P}^1$ (resp. $\mathbb{P}^2$). In this case the Gysin sequence [85] gives:

$$\begin{gathered} H^0(M^{111}) = H^7(M^{111}) = \mathbb{Z}, \\ 0 \longrightarrow H^1(M^{111}) \longrightarrow \mathbb{Z} \xrightarrow{c_1} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^2(M^{111}) \longrightarrow 0, \\ 0 \longrightarrow H^3(M^{111}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{c_1} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H^4(M^{111}) \longrightarrow 0, \\ 0 \longrightarrow H^5(M^{111}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{c_1} \mathbb{Z} \longrightarrow H^6(M^{111}) \longrightarrow 0. \end{gathered} \tag{4.4.31}$$

The first c_1 sends $1 \in H^0(M_a)$ to $c_1 \in H^2(M_a)$. Its kernel is zero, and its image is $\mathbb{Z}$. Accordingly, $H^2(M^{111}) = \mathbb{Z} \cdot \pi^*(\omega_1 + \omega_2)$. The second c_1 sends $(\omega_1, \omega_2) \in \mathbb{Z} \oplus \mathbb{Z} = H^2(M_a)$ to $(3\omega_1\omega_2, 2\omega_1\omega_2 + 3\omega_2^2) \in \mathbb{Z} \oplus \mathbb{Z} = H^4(M_a)$. Its kernel vanishes and therefore $H^3(M^{111}) = 0$. Its cokernel is $\mathbb{Z}_9 = H^4(M^{111})$ generated by $\pi^*(\omega_1\omega_2 + \omega_2^2)$. Finally, the last c_1 sends $\omega_1\omega_2$ and $\omega_2^2 \in H^4(M_a) = \mathbb{Z} \oplus \mathbb{Z}$ respectively to $3\omega_1\omega_2^2$ and $2\omega_1\omega_2^2 \in H^6(M_a)$. This map is surjective, so $H^6(M^{111}) = 0$ and its kernel is generated by $\beta = -2\omega_1\omega_2 + 3\omega_2^2$. Hence $H^5(M^{111}) = \mathbb{Z} \cdot \alpha$, with $\pi_*\alpha = \beta$.

Explicit description of the $U(1)$ fibration for M^{111}

We proceed next to an explicit description of the fibration structure of M^{111} as a $U(1)$-bundle over $\mathbb{P}^2 \times \mathbb{P}^1$. We construct an atlas of local trivializations and we give the appropriate transition functions. This is important for our discussion of the supersymmetric cycles leading to the baryon states.

We take $\tau \in [0, 4\pi)$ as a local coordinate on the fibre and $(\widetilde{\theta}, \widetilde{\phi})$ as local coordinates on $\mathbb{P}^1 \simeq S^2$. To describe $\mathbb{P}^2$ we have to be a little bit careful. $\mathbb{P}^2$ can be covered by the three patches $W_\alpha \simeq \mathbb{C}^2$ in which one of the three homogeneous coordinates, U_α, does not vanish. The set not covered by one of these W_α is homeomorphic to S^2. We choose to parametrize W_3 as in [86]:

$$\left\{ \begin{array}{l} \zeta^1 = U_1/U_3 = \tan\mu\, \cos(\theta/2)\, e^{i(\psi+\phi)/2} \\ \zeta^2 = U_2/U_3 = \tan\mu\, \sin(\theta/2)\, e^{i(\psi-\phi)/2} \end{array} \right. , \tag{4.4.32}$$

where

$$\left\{ \begin{array}{l} \mu \in (0, \pi/2) \\ \theta \in (0, \pi) \\ 0 \le (\psi + \phi) \le 4\pi \\ 0 \le (\psi - \phi) \le 4\pi \end{array} \right. . \tag{4.4.33}$$

These coordinates cover the whole $W_3 \simeq \mathbb{C}^2$ except for the trivial coordinate singularities $\mu = 0$ and $\theta = 0, \pi$. Furthermore θ and ϕ can be extended to the complement of W_3. Indeed, the ratio

$$z = \zeta^1/\zeta^2 = \tan^{-1}(\theta/2)\, e^{i\phi} \tag{4.4.34}$$

is well defined in the limit $\mu \to \pi/2$ and it constitutes the usual stereographic map of S^2 onto the complex plane (see the next discussion of Q^{111} and in particular figure 4.4).

We must be careful in treating some one-forms near the coordinate singularities. In particular, $d\psi$ and $d\phi$ are not well defined on the three S^2 which are not covered by one of the patches W_α: $\{\mu = \pi/2\}$, $\{\theta = 0\}$ and $\{\theta = \pi/2\}$ (see figure 4.3.) Actually, except for the three points of these spheres that are covered by only one patch ($\{\mu = 0\} \in W_3$, $\{\mu = \pi/2, \theta = 0\} \in W_1$, $\{\mu = \pi/2, \theta = \pi\} \in W_2$), one particular combination of $d\psi$ and $d\phi$ survives, as it is illustrated in table (4.4.35).

coordinate singularity	regular one − form	singular one − forms
$\theta = 0$	$d\psi + d\phi$	$\alpha d\psi + \beta d\phi$ ($\alpha \neq \beta$)
$\theta = \pi$	$d\psi - d\phi$	$\alpha d\psi - \beta d\phi$ ($\alpha \neq \beta$)
$\mu = \pi/2$	$d\phi$	$\alpha d\psi$

(4.4.35)

The singular one-forms become well defined if we multiply them by a function having a double zero at the coordinate singularities.

We come now to the description of the fibre bundle M^{111}. We cover the base $\mathbb{P}^2 \times \mathbb{P}^1$ with six open charts $\mathcal{U}_{\alpha\pm} = W_\alpha \times H_\pm$ ($\alpha = 1, 2, 3$) on which we can define a local fibre coordinate $\tau_{\alpha\pm} \in [0, 4\pi)$. The transition functions are given by:

$$\left\{ \begin{array}{ll} \tau_{1\beta} = \tau_{3\gamma} - 3(\psi + \phi) + 2(\beta - \gamma)\widetilde{\phi}\,, & (\beta, \gamma = \pm 1) \\ \tau_{1\beta} = \tau_{2\gamma} - 6\phi + 2(\beta - \gamma)\widetilde{\phi}\,. & \end{array} \right. \tag{4.4.36}$$

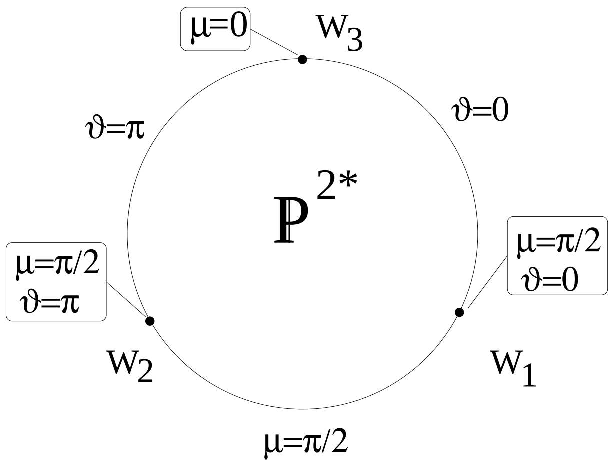


Figure 4.3: Schematic representation of the atlas on $\mathbb{P}^2$. The three patches W_α cover the open ball and part of the boundary circle, which constitutes the set of coordinate singularities. This latter is made of three S^2's: $\{\theta = 0\}$, $\{\theta = \pi\}$ and $\{\mu = \pi/2\}$, which touch each other at the three points marked with a dot. Each W_α covers the whole $\mathbb{P}^2$ except for one of the spheres (for example, W_3 does not cover $\{\mu = \pi/2\}$). The three *most singular* points are covered by only one patch (for example, $\{\mu = 0\}$ is covered by the only W_3).

On this principal fibre bundle we can easily introduce a $U(1)$ Lie algebra valued connection which, on the various patches of the base space, is described by the following one–forms:

$$\begin{cases} \mathcal{A}_{1\pm} = -\frac{3}{2}(\cos 2\mu+1)(d\psi + d\phi) - \frac{3}{2}(\cos 2\mu-1)(\cos\theta-1)d\phi + 2(\pm 1-\cos\widetilde{\theta})d\widetilde{\phi}\,, \\ \mathcal{A}_{2\pm} = -\frac{3}{2}(\cos 2\mu+1)(d\psi - d\phi) - \frac{3}{2}(\cos 2\mu-1)(\cos\theta+1)d\phi + 2(\pm 1-\cos\widetilde{\theta})d\widetilde{\phi}\,, \\ \mathcal{A}_{3\pm} = -\frac{3}{2}(\cos 2\mu - 1)(d\psi + \cos\theta d\phi) + 2(\pm 1 - \cos\widetilde{\theta})d\widetilde{\phi}\,. \end{cases} \tag{4.4.37}$$

Due to (4.4.36), the one-form $(d\tau - \mathcal{A})$ is a global angular form [85]. It can then be taken as the 7-th vielbein of the following $SU(3) \times SU(2) \times U(1)$ invariant metric on M^{111}:

$$ds^2_{M^{111}} = c^2(d\tau - \mathcal{A})^2 + ds^2_{\mathbb{P}^2} + ds^2_{\mathbb{P}^1}. \tag{4.4.38}$$

The one-form $\mathcal{A}$ is the connection of the Hodge-Kähler bundle on $\mathbb{P}^2 \times \mathbb{P}^1$.

Einstein Metric

The Einstein metric on the homogeneous space M^{111} can be written in terms of the vielbein given in chapter 3 and found in [7]. However, the same metric can be expressed (see [87]) in the coordinate frame we have just utilized to describe the fibration structure and which is convenient for our discussion of the supersymmetric 5–cycles. In this frame it is

$$ds^2_{M^{111}} = \frac{3}{32\Lambda}\left[d\tau - 3\sin^2\mu\ (d\psi + \cos\theta d\phi) + 2\cos\widetilde{\theta}d\widetilde{\phi}\right]^2 +$$

$$
\begin{aligned}
&+ \quad \frac{9}{2\,\Lambda}\left[d\mu^2 + \frac{1}{4}\sin^2\mu\cos^2\mu^2\left(d\psi + \cos\theta d\phi\right)^2 + \right.\\
&+ \quad \left.\frac{1}{4}\sin^2\mu\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] + \frac{3}{4\Lambda}\left(d\widetilde{\theta}^2 + \sin^2\widetilde{\theta} d\widetilde{\phi}^2\right) . \qquad (4.4.39)
\end{aligned}
$$

The second and the third addenda are the $\mathbb{P}^2$ and S^2 metric on the base manifold of the $U(1)$ fibration, while the first term is the fibre metric. In other words, one recognizes the structure of the metric anticipated in (4.4.38). The parameter Λ appearing in the metric (4.4.39) is the internal cosmological constant defined by eq. (4.4.16).

The baryonic 5–cycles of M^{111} and their volume

As we saw above, the relevant homology group of M^{111} for the calculation of the baryonic masses is

$$
H_5(M^{111}, \mathbb{R}) = \mathbb{R} . \qquad (4.4.40)
$$

Let us consider the following two five-cycles, belonging to the same homology class:

$$
\mathcal{C}^1 : \begin{cases} \widetilde{\theta} = \widetilde{\theta}_0 = const \\ \widetilde{\phi} = \widetilde{\phi}_0 = const \end{cases} , \qquad (4.4.41)
$$

$$
\mathcal{C}^2 : \begin{cases} \theta = \theta_0 = const \\ \phi = \phi_0 = const \end{cases} . \qquad (4.4.42)
$$

The two representatives (4.4.41, 4.4.42) are distinguished by their different stability subgroups which we calculate in the next subsection.

Volume of the 5–cycles

The volume of the cycles (4.4.41, 4.4.42) is easily computed by pulling back the metric (4.4.39) on $\mathcal{C}^1$ and $\mathcal{C}^2$, that have the topology of a $U(1)$-bundle over $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ respectively:

$$
\mathrm{Vol}(\mathcal{C}^1) = \oint_{\mathcal{C}^1} \sqrt{g_1} = 9\,(8\Lambda/3)^{-5/2} \int \sin^3\mu\cos\mu\sin\theta\, d\tau d\mu d\psi d\theta d\phi = \frac{9\pi^3}{2}\left(\frac{3}{2\Lambda}\right)^{5/2} \qquad (4.4.43)
$$

$$
\mathrm{Vol}(\mathcal{C}^2) = \oint_{\mathcal{C}^2} \sqrt{g_2} = 6\,(8\Lambda/3)^{-5/2} \int \sin\mu\cos\mu\sin\widetilde{\theta}\, d\tau d\mu d\psi d\widetilde{\theta} d\widetilde{\phi} = 6\pi^3\left(\frac{3}{2\Lambda}\right)^{5/2} . \qquad (4.4.44)
$$

The volume of M^{111} is instead given by

$$
\begin{aligned}
\mathrm{Vol}(M^{111}) = \oint_{M^{111}} \sqrt{g} = 18\,(8\Lambda/3)^{-7/2} \int \sin^3\mu\cos\mu\sin\theta\sin\widetilde{\theta}\, d\tau d\mu d\psi d\theta d\phi d\widetilde{\theta} d\widetilde{\phi} + \\
= \frac{27\pi^4}{2\Lambda}\left(\frac{3}{2\Lambda}\right)^{5/2} .
\end{aligned} \qquad (4.4.45)
$$

The results (4.4.43, 4.4.44, 4.4.45) can be inserted into the general formula (4.4.29) to calculate the conformal weights (or energy labels) of five-branes wrapped on the cycles $\mathcal{C}^1$ and $\mathcal{C}^2$. We obtain:

$$\begin{cases} E_0(\mathcal{C}^1) = N/3 \\ \\ E_0(\mathcal{C}^2) = 4N/9 \end{cases} . \tag{4.4.46}$$

As stated above, the result (4.4.46) is essential in proving that the conformal weight of the elementary world–volume fields V^A, U^i are

$$h\left[V^4\right] = 1/3 \quad , \quad h\left[U^i\right] = 4/9 \tag{4.4.47}$$

respectively. To reach such a conclusion we need to identify the states obtained by wrapping the five–brane on $\mathcal{C}^1, \mathcal{C}^2$ with operators in the flavour representations $M_1 = 0, M_2 = 0, J = N/2$ and $M_1 = N, M_2, J = 0$, respectively. This conclusion is reached by studying the stability subgroups of the supersymmetric 5–cycles.

This matches with the previous result (4.3.6) on the spectrum of chiral operators, which are predicted of the form

$$\mathrm{Tr}\left(U^3 V^2\right)^k \tag{4.4.48}$$

and should have conformal weight $E = 2k$. Indeed, we have

$$3 \times \frac{4}{9} + 2 \times \frac{1}{3} = 2\,!!! \tag{4.4.49}$$

The flavour representations of the baryons

To find the flavour representations of these non–perturbative states we follow an argument introduced by Witten [79], where they are found by studying the stability subgroups of the five–cycles $H\left(\mathcal{C}^i\right)$. As shown in [79], the collective degrees of freedom c of the wrapped 5–brane soliton live on the coset manifold $G/H(\mathcal{C}^i)$, where G is the isometry group of X_7. The wave–function $\Psi(c)$ of the soliton must be expanded in harmonics on $G/H(\mathcal{C}^i)$ characterized by having charge N under the baryon number $U(1)_B \subset H(\mathcal{C}^i)$. Minimizing the energy operator (the laplacian) on such harmonics one obtains the corresponding G representation and hence the flavour assignment of the baryon.

Let us now consider the stability subgroups

$$H(\mathcal{C}_i) \subset G = SU(3) \times SU(2) \times U(1) \tag{4.4.50}$$

of the two cycles (4.4.41, 4.4.42). Let us begin with the first cycle defined by (4.4.41). As we have previously said, this is the restriction of the $U(1)$-fibration to $\mathbb{P}^2 \times \{p\}$, p being a point of $\mathbb{P}^1$. Hence, the stability subgroup of the cycle $\mathcal{C}^1$ is:

$$H\left(\mathcal{C}^1\right) = SU(3) \times U(1)_R \times U(1)_{B,1} \tag{4.4.51}$$

where $U(1)_R$ is the R–symmetry $U(1)$ appearing as a factor in $SU(3) \times SU(2) \times U(1)_R$ while $U(1)_{B,1} \subset SU(2)$ is a maximal torus.

Turning to the case of the second cycle (4.4.42), which is the restriction of the $U(1)$-bundle to the product of a hyperplane of $\mathbb{P}^2$ and $\mathbb{P}^1$, its stabilizer is

$$H\left(\mathcal{C}^2\right) = SU(2) \times U(1)_{B,2} \times SU(2) \times U(1)_R, \tag{4.4.52}$$

where $SU(2) \times U(1)_R$ is the group appearing as a factor in $SU(3) \times SU(2) \times U(1)_R$, $U(1)_{B,2} \subset SU(3)$ is the subgroup generated by $h_1 = \mathrm{diag}(1,-1,0)$ and $SU(2) \times U(1)_{B,2} \subset SU(3)$ is the stabilizer of the first basis vector of $\mathbb{C}^3$.

Following the procedure introduced by Witten in [79] we should now quantize the *collective coordinates* of the non–perturbative baryon state obtained by *wrapping* the five–brane on the 5–cycles we have been discussing. As explained in Witten's paper this leads to quantum mechanics on the homogeneous manifold $G/H(\mathcal{C})$. In our case the collective coordinates of the baryon live on the following spaces:

$$\text{space of collective coordinates} \quad \rightarrow \quad \frac{G}{H(\mathcal{C})} = \begin{cases} \frac{SU(2)}{U(1)_{B,1}} \simeq \mathbb{P}^1 & \text{for } \mathcal{C}^1 \\ \frac{SU(3)}{SU(2)\times U(1)_{B,2}} \simeq \mathbb{P}^2 & \text{for } \mathcal{C}^2 \end{cases} . \tag{4.4.53}$$

The wave function Ψ (collec. coord.) is in Witten's phrasing a section of a line bundle of degree N. This happens because the baryon has *baryon number* N, namely it has charge N under the additional massless vector multiplet that is associated with a harmonic 2–form and appears in the Kaluza Klein spectrum since $\mathrm{dim} H_2(M^{111}) = 1 \neq 0$. These are the Betti multiplets mentioned in Section 4.4. Following Witten's reasoning there is a morphism

$$\mu^i : \quad U(1)_{Baryon} \hookrightarrow H(\mathcal{C}^i) \quad i = 1,2 \tag{4.4.54}$$

of the non perturbative baryon number group into the stability subgroup of the 5–cycle. Clearly the image of such a morphism must be a $U(1)$–factor in $H(\mathcal{C})$ that has a non trivial action on the collective coordinates of the baryons. Clearly in the case of our two baryons we have:

$$\mathrm{Im}\,\mu^i = U(1)_{B,i} \quad i = 1,2\,. \tag{4.4.55}$$

The name given to these groups anticipated the conclusions of such an argument.

Translated into the language of harmonic analysis, Witten's statement that the baryon wave function should be a section of a line bundle with degree N means that we are supposed to consider harmonics on $G/H(\mathcal{C})$ which, rather than being scalars of $H(\mathcal{C})$, are in the 1–dimensional representation of $U(1)_B$ with charge N. According to the general rules of harmonic analysis (see chapter 3) we are supposed to collect all the representations of G whose reduction with respect to $H(\mathcal{C})$ contains the prescribed representation of $H(\mathcal{C})$. In the case of the first cycle, in view of eq. (4.4.51) we want all representations of $SU(2)$ that contain the state $J^{(3)} = N$. Indeed the generator of $U(1)_{B,1}$ can always be regarded as the third component of angular momentum by means of a change of basis. The representations with this property are those characterized by:

$$2J = N + 2k, \qquad k \geq 0. \tag{4.4.56}$$

Since the laplacian on $G/H(\mathcal{C})$ has eigenvalues proportional to the Casimir

$$\Box_{SU(2)/U(1)} = \text{const} \times J(J+1), \tag{4.4.57}$$

the harmonic satisfying the constraint (4.4.56) and with minimal energy is just that with

$$2J = N. \tag{4.4.58}$$

This shows that under the flavour group the baryon associated with the first cycle is neutral with respect to $SU(3)$ and transforms in the N–times symmetric representation of $SU(2)$. This perfectly matches, on the superconformal field theory side, with our candidate operator (4.4.3).

Equivalently the choice of the representation $2J = N$ corresponds with the identification of the baryon wave–function with a *holomorphic section (=zero mode)* of the $U(1)$–bundle under consideration, i.e. with a section of the corresponding line bundle. Indeed such a line bundle is, by definition, constructed over $\mathbb{P}^1$ and declared to be of degree N, hence it is $\mathcal{O}_{\mathbb{P}^1}(N)$. Representation-wise a section of $\mathcal{O}_{\mathbb{P}^1}(N)$ is just an element of the $J = N/2$ representation, namely it is the N times symmetric of $SU(2)$.

Let us now consider the case of the second cycle. Here the same reasoning instructs us to consider all representations of $SU(3)$ which, reduced with respect to $U(1)_{B,2}$, contain a state of charge N. Moreover, directly aiming at zero mode, we can assign the baryon wave–function to a holomorphic section of a line bundle on $\mathbb{P}^2$, which must correspond to characters of the parabolic subgroup $SU(2) \times U(1)_{B,2}$. As before the degree N of this line bundle uniquely characterizes it as $\mathcal{O}(N)$. In the language of Young tableaux, the corresponding $SU(3)$ representation is

$$M_1 = 0\,;\; M_2 = N, \tag{4.4.59}$$

i.e. the representation of this baryon state is the N–time symmetric of the dual of $SU(3)$ and this perfectly matches with the complex conjugate of the candidate conformal operator 4.4.2. In other words we have constructed the antichiral baryon state. The chiral one obviously has the same conformal dimension.

These 5–cycles are supersymmetric

The 5–cycles we have been considering in the above subsections have to be supersymmetric in order for the conclusions we have been drawing to be correct. Indeed all our arguments have been based on the assumption that the 5–brane wrapped on such cycles is a BPS–state. This is true if the 5–brane action localized on the cycle is κ–supersymmetric.

The κ-symmetry projection operator for a five-brane is

$$P_{\pm} = \frac{1}{2}\left(\mathbb{1} \pm \mathrm{i}\,\frac{1}{5!\,\sqrt{g}}\epsilon^{\alpha\beta\gamma\delta\varepsilon}\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P\partial_\delta X^Q\partial_\varepsilon X^R\,\Gamma_{MNPQR}\right)\,, \tag{4.4.60}$$

where the functions $X^M(\sigma^\alpha)$ define the embedding of the five-brane into the eleven dimensional spacetime, and $\sqrt{g}$ is the square root of the determinant of the induced metric on the brane. The gamma matrices Γ_{MNPQR}, defining the spacetime spinorial structure, are the pullback through the vielbein of the constant gamma matrices Γ_{ABCDE} satisfying the standard Clifford algebra:

$$\Gamma_{MNPQR} = e^A_M e^B_N e^C_P e^D_Q e^E_R \Gamma_{ABCDE}\,. \tag{4.4.61}$$

A possible choice of vielbein for $\mathcal{C}(M^{111}) \times \mathcal{M}_3$, namely the product of the metric cone

over M^{111} times three dimensional Minkowski space is the following one:

$$\left\{\begin{array}{rcl} e^1 & = & \frac{1}{2\sqrt{2}}\, r\, d\widetilde{\theta} \\ e^2 & = & \frac{1}{2\sqrt{2}}\, r \sin\widetilde{\theta} d\widetilde{\phi} \\ e^3 & = & \frac{1}{8}\, r \left(d\tau + 3\sin^2\mu (d\psi + \cos\theta d\phi) + 2\cos\widetilde{\theta} d\widetilde{\phi} \right) \\ e^4 & = & \frac{\sqrt{3}}{2}\, r\, d\mu \\ e^5 & = & \frac{\sqrt{3}}{4}\, r \sin\mu\cos\mu\, (d\psi + \cos\theta d\phi) \\ e^6 & = & \frac{\sqrt{3}}{4}\, r\sin\mu\, (\sin\psi d\theta - \cos\psi \sin\theta d\phi) \\ e^7 & = & \frac{\sqrt{3}}{4}\, r\sin\mu\, (\cos\psi d\theta + \sin\psi\sin\theta d\phi) \\ e^8 & = & dr \\ e^9 & = & dx^1 \\ e^{10} & = & dx^2 \\ e^0 & = & dt \end{array}\right. . \tag{4.4.62}$$

In these coordinates the embedding equations of the two cycles (4.4.41), (4.4.42) are very simple, so we have

$$\frac{1}{5!}\epsilon^{\alpha\beta\gamma\delta\varepsilon}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \partial_\delta X^Q \partial_\varepsilon X^R\, \Gamma_{MNPQR} = \left\{ \begin{array}{l} \Gamma_{\tau\mu\theta\psi\phi} \\ \Gamma_{\tau\mu\widetilde{\theta}\psi\widetilde{\phi}} \end{array}\right. , \tag{4.4.63}$$

for $\mathcal{C}^1$ and $\mathcal{C}^2$ respectively. By means of the vielbein (4.4.62) these gamma matrices are immediately computed:

$$\left\{ \begin{array}{r} \Gamma_{\tau\mu\theta\psi\phi} = \left(\frac{3}{32}\right)^2 r^5 \sin^3\mu\cos\mu\sin\theta\,\Gamma_{34567} \\ \Gamma_{\tau\mu\widetilde{\theta}\psi\widetilde{\phi}} = \frac{3}{512} r^5 \sin\mu\cos\mu\sin\widetilde{\theta}\,\Gamma_{31245} \end{array}\right. , \tag{4.4.64}$$

while the square root of the determinant of the metric on the two cycles is easily seen to be

$$\left\{ \begin{array}{r} \sqrt{g_1} = \left(\frac{3}{32}\right)^2 r^5 \sin^3\mu\cos\mu\sin\theta \\ \sqrt{g_1} = \frac{3}{512} r^5 \sin\mu\cos\mu\sin\widetilde{\theta} \end{array}\right. . \tag{4.4.65}$$

So, for both cycles, the κ-symmetry projector (4.4.60) reduces to the projector of a five dimensional hyperplane embedded in flat spacetime:

$$P_\pm = \left\{ \begin{array}{l} \frac{1}{2}\,(\mathbb{1} \pm \mathrm{i}\,\Gamma_{34567}) \\ \frac{1}{2}\,(\mathbb{1} \pm \mathrm{i}\,\Gamma_{31245}) \end{array}\right. . \tag{4.4.66}$$

The important thing to check is that the projectors (4.4.66) are non–zero on the two Killing spinors of the space $\mathcal{C}(M^{111})\times\mathcal{M}_3$. Indeed, this latter has not 32 preserved supersymmetries, rather it has only 8 of them. In order to avoid long and useless calculations we just argue as follows. Using the gamma–matrix basis of [7], the Killing spinors are already known. We have:

$$\begin{array}{rclcrcl} \Gamma_0 & = & \gamma_0 \otimes \mathbf{1}_{8\times 8} & ; & \Gamma_8 & = & \gamma_1 \otimes \mathbf{1}_{8\times 8} \\ \Gamma_9 & = & \gamma_2 \otimes \mathbf{1}_{8\times 8} & ; & \Gamma_{10} & = & \gamma_3 \otimes \mathbf{1}_{8\times 8} \\ \Gamma_i & = & \gamma_5 \otimes \tau_i & (i=1,\dots,7) & & & \end{array} \tag{4.4.67}$$

where $\gamma_{0,1,2,3}$ are the usual 4×4 gamma matrices in four–dimensional space–time, while τ_i are the 8×8 gamma–matrices satisfying the $SO(7)$ Clifford algebra in the form: $\{\tau_i\,,\,\tau_j\} =$

$-\delta_{ij}$. For these matrices we take the representation given in the Appendix of [7], which is well adapted to the intrinsic description of the M^{111} metric through Maurer–Cartan forms. In this basis the Killing spinors were calculated in [7] and have the following form:

$$\text{Killing spinors} \quad = \quad \epsilon(x) \otimes \eta \quad ; \quad \eta = \left(\begin{array}{c} \mathbf{0} \\ \hline \mathbf{u} \\ \hline \mathbf{0} \\ \hline \epsilon \mathbf{u}^\star \end{array} \right) , \tag{4.4.68}$$

where

$$\mathbf{u} \quad = \quad \left(\begin{array}{c} a + \mathrm{i}b \\ 0 \end{array} \right) \quad ; \quad \epsilon \mathbf{u}^\star = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \mathbf{u}^\star = \left(\begin{array}{c} 0 \\ -a + \mathrm{i}b \end{array} \right) \tag{4.4.69}$$

and where the 8–component spinor was written in 2–component blocks.

In the same basis, using notations of [7], we have:

$$\Gamma_{34567} \quad = \quad \gamma_5 \otimes U_8 \, U_4 \, U_5 \, U_6 \, U_7 \otimes \sigma_3 \quad = \quad \mathrm{i} \, \gamma_5 \otimes \left(\begin{array}{c|c|c|c} -\mathbf{1}_{2\times 2} & 0 & 0 & 0 \\ \hline 0 & \mathbf{1}_{2\times 2} & 0 & 0 \\ \hline 0 & 0 & \mathbf{1}_{2\times 2} & 0 \\ \hline 0 & 0 & 0 & -\mathbf{1}_{2\times 2} \end{array} \right),$$

$$\Gamma_{31245} \quad = \quad \gamma_5 \otimes \mathrm{i} U_8 \, U_4 \, U_5 \otimes \mathbf{1} \quad = \quad \mathrm{i} \, \gamma_5 \otimes \left(\begin{array}{c|c|c|c} \sigma_3 & 0 & 0 & 0 \\ \hline 0 & \sigma_3 & 0 & 0 \\ \hline 0 & 0 & \sigma_3 & 0 \\ \hline 0 & 0 & 0 & \sigma_3 \end{array} \right). \tag{4.4.70}$$

As we see, by comparing eq. (4.4.66) with eq. (4.4.68) and (4.4.70), the κ–supersymmetry projector reduces for both cycles to a chirality projector on the 4–component space–time part $\epsilon(x)$. As such, the κ–supersymmetry projector always admits non vanishing eigenstates implying that the cycle is supersymmetric. The only flaw in the above argument is that the Killing spinor (4.4.68) was determined in [7] using as vielbein basis the suitably rescaled Maurer–Cartan forms $\mathcal{B}^3$, $\mathcal{B}^m$, $(m = 1, 2)$ and $\mathcal{B}^A$, $(A = 4, 5, 6, 7)$ (see chapter 3). Our choice (4.4.62) does not correspond to the same vielbein basis. However, a little inspection shows that it differs only by some $SO(4)$ rotation in the space of $\mathbb{P}^2$ vielbein $4, 5, 6, 7$. Hence we can turn matters around and ask what happens to the Killing spinor (4.4.68) if we apply an $SO(4)$ rotation in the directions $4, 5, 6, 7$. It suffices to check the form of the gamma–matrices $[\tau_A \, , \, \tau_B]$ which are the generators of such rotations. Using again the Appendix of [7] we see that such $SO(4)$ generators are of the form

$$\mathrm{i} \left(\begin{array}{c|c|c|c} \sigma_i & 0 & 0 & 0 \\ \hline 0 & \sigma_i & 0 & 0 \\ \hline 0 & 0 & \sigma_i & 0 \\ \hline 0 & 0 & 0 & \sigma_i \end{array} \right) \quad \text{or} \quad \mathrm{i} \left(\begin{array}{c|c|c|c} \sigma_i & 0 & 0 & 0 \\ \hline 0 & -\sigma_i & 0 & 0 \\ \hline 0 & 0 & \sigma_i & 0 \\ \hline 0 & 0 & 0 & -\sigma_i \end{array} \right), \tag{4.4.71}$$

so that the $SO(4)$ rotated Killing spinor is of the same form as in eq.(4.4.68) with, however, $\mathbf{u}$ replaced by $\mathbf{u}' = A\mathbf{u}$ where $A \in SU(2)$. It is obvious that such an $SU(2)$ transformation does not alter our conclusions. We can always decompose $\mathbf{u}'$ into σ_3 eigenstates and associate the σ_3–eigenvalue with the chirality eigenvalue, so as to satisfy the κ–supersymmetry projection. Hence, our 5–cycles are indeed supersymmetric.

4.4.3 The case of Q^{111}

Cohomology of Q^{111}

As for the cohomology [85], the first Chern class of L is $c_1 = \omega_1 + \omega_2 + \omega_3$, where ω_i are the generators of the second cohomology group of the $\mathbb{P}^1$'s. Reasoning as for M^{111}, one gets

$$\begin{aligned} H^1(Q^{111}, \mathbb{Z}) = H^3(Q^{111}, \mathbb{Z}) = H^6(Q^{111}, \mathbb{Z}) = 0, \\ H^2(Q^{111}, \mathbb{Z}) = \mathbb{Z} \cdot \omega_1 \oplus \mathbb{Z} \cdot \omega_2, \\ H^4(Q^{111}, \mathbb{Z}) = \mathbb{Z}_2 \cdot (\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3), \\ H^5(Q^{111}, \mathbb{Z}) = \mathbb{Z} \cdot \alpha \oplus \mathbb{Z} \cdot \beta, \end{aligned} \tag{4.4.72}$$

where $\pi_*\alpha = \omega_1\omega_2 - \omega_1\omega_3$, $\pi_*\beta = \omega_1\omega_2 - \omega_2\omega_3$ and the pullbacks are left implicit.

Explicit description of the $U(1)$ fibration for Q^{111}

The coset space Q^{111} is a $U(1)$-fibre bundle over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \simeq S^2 \times S^2 \times S^2$. We can parametrize the base manifold with polar coordinates (θ_i, ϕ_i), $i = 1, 2, 3$. We cover the base with eight coordinate patches, $H_{\alpha\beta\gamma}$ $(\alpha, \beta, \gamma = \pm 1)$ and choose local coordinates for the fibre, $\psi_{\alpha\beta\gamma} \in [0, 4\pi)$. Every patch is the product of three open sets, $H^i_\pm$, each one describing a coordinate patch for a single two-sphere, as indicated in fig. 4.4:

$$H_{\alpha\beta\gamma} = H^1_\alpha \times H^2_\beta \times H^3_\gamma. \tag{4.4.73}$$

To describe the total space we have to specify the transition maps for ψ on the intersec-

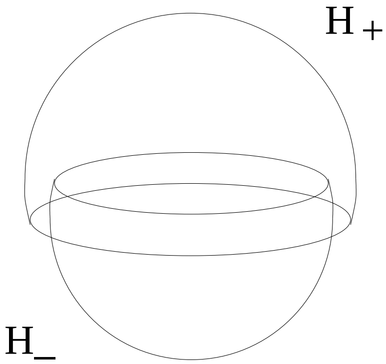


Figure 4.4: Two coordinate patches for the sphere. They constitute the base for a local trivialization of a fibre bundle on S^2. Each patch covers only one of the poles, where the coordinates (θ, ϕ) are singular.

tions of the patches. These maps for the generic Q^{pqr} space are

$$\psi_{\alpha_1\beta_1\gamma_1} = \psi_{\alpha_2\beta_2\gamma_2} + p(\alpha_1 - \alpha_2)\phi_1 + q(\beta_1 - \beta_2)\phi_2 + r(\gamma_1 - \gamma_2)\phi_3\,. \tag{4.4.74}$$

For example, in the case of interest, Q^{111}, we have

$$\psi_{+-+} = \psi_{++-} - 2\phi_2 + 2\phi_3\,. \tag{4.4.75}$$

We note that these maps are well defined, being all the ψ's and ϕ's defined modulo 4π and 2π respectively.

It is important to note that θ and ϕ are clearly not good coordinates for the whole S^2. The most important consequence of this fact is that the one-form $d\phi$ is not extensible to the poles. To extend it to one of the poles, $d\phi$ has to be multiplied by a function which has a double zero on that pole, such as $\sin^2 \frac{\theta}{2}\, d\phi$.

We can define a $U(1)$-connection $\mathcal{A}$ on the base $S^2 \times S^2 \times S^2$ by specifying it on each patch $H_{\alpha\beta\gamma}$ [10]:

$$\mathcal{A}_{\alpha\beta\gamma} = (\alpha - \cos\theta_1)d\phi_1 + (\beta - \cos\theta_2)d\phi_2 + (\gamma - \cos\theta_3)d\phi_3 \,. \tag{4.4.76}$$

Because of the fibre-coordinate transition maps (4.4.74), the one-form $(d\psi - \mathcal{A})$ is globally well defined on Q^{111}. In other words the different one-forms $(d\psi_{\alpha\beta\gamma} - \mathcal{A}_{\alpha\beta\gamma})$ defined on the corresponding $H_{\alpha\beta\gamma}$, coincide on the intersections of the patches. We can therefore define an $SU(2)^3 \times U(1)$-invariant metric on the total space by:

$$ds^2_{Q^{111}} = c^2(d\psi - \mathcal{A})^2 + a^2 ds^2_{S^2\times S^2\times S^2} \,. \tag{4.4.77}$$

The Einstein metric of this family is given by

$$ds^2_{Q^{111}} = \frac{3}{8\Lambda}(d\psi - \mathcal{A})^2 + \frac{3}{4\Lambda}\sum_{i=1}^{3}\left(d\theta_i^2 + \sin^2\theta_i \, d\phi_i^2\right) \,, \tag{4.4.78}$$

where Λ is the compact space cosmological constant defined in eq.(4.4.16). The Einstein metric (4.4.78) was originally found in [60], using the intrinsic geometry of coset manifolds and using Maurer–Cartan forms. An explicit form was also given using stereographic coordinates on the three S^2. In the coordinate form of eq. (4.4.78) the Einstein metric of Q^{111} was later given in [88].

The baryonic 5–cycles of Q^{111} and their volume

The relevant homology group of Q^{111} for the calculation of the baryonic masses is

$$H_5(Q^{111}, \mathbb{R}) = \mathbb{R}^2 \,. \tag{4.4.79}$$

Three (dependent) five-cycles spanning $H_5(Q^{111})$ are the restrictions of the $U(1)$-fibration to the product of two of the three $\mathbb{P}^1$'s. Using the above metric (4.4.78) one easily computes the volume of these cycles. For instance

$$\mathrm{Vol(cycle)} = \oint_{\pi^{-1}(\mathbf{P}^1_1\times\mathbf{P}^1_2)} \left(\frac{3}{8\Lambda}\right)^{5/2} 4\sin\theta_1\, \sin\theta_2\; d\theta_1\, d\theta_2\, d\phi_1\, d\phi_2\, d\psi = \frac{\pi^3}{4}\left(\frac{6}{\Lambda}\right)^{5/2} \,. \tag{4.4.80}$$

The volume of the whole space Q^{111} is

$$\mathrm{Vol(Q^{111})} = \oint_{Q^{111}} \left(\frac{3}{8\Lambda}\right)^{7/2} 8\prod_{i=1}^{3}\sin\theta_i\, d\theta_i d\phi_i\, d\psi = \frac{\pi^4}{8}\left(\frac{6}{\Lambda}\right)^{7/2} \,. \tag{4.4.81}$$

[10] It is worth noting that the connection $\mathcal{A}$ is chosen to be well defined on the coordinate singularities of each patch, i.e. on the product of the three S^2 poles covered by the patch.

Just as in the M^{111} case, inserting the above results (4.4.80, 4.4.81) into the general formula (4.4.29) we obtain the conformal weight of the baryon operator corresponding to the five-brane wrapped on this cycle:

$$E_0 = \frac{N}{3} \,. \tag{4.4.82}$$

The other two cycles can be obtained from this by permuting the role of the three $\mathbb{P}^1$'s and their volume is the same. This fact agrees with the symmetry which exchanges the fundamental fields A, B and C of the conformal theory, or the three gauge groups $SU(N)$. Indeed, naming $SU(2)_i$ ($i = 1, 2, 3$) the three $SU(2)$ factors appearing in the isometry group of Q^{111}, the stability subgroup of the first of the cycles described above is

$$\begin{array}{rcl} H(\mathcal{C}^1) & = & SU(2)_1 \times SU(2)_2 \times U(1)_{B,3} \\ U(1)_{B,3} & \subset & SU(2)_3 \end{array} \tag{4.4.83}$$

so that the collective coordinates of the baryon state live on $\mathbb{P}^1 \simeq SU(2)_3/U(1)_{B,3}$. This result is obtained by an argument completely analogous to that used in the analysis of M^{111} 5–cycles and leads to a completely analogous conclusion. The baryon state is in the $J^{(1)} = 0$, $J^{(2)} = 0$, $J^{(3)} = N/2$ flavour representation. In the conformal field theory the corresponding baryon operator is the chiral field (4.4.6) and the result (4.4.82) implies that the conformal weight of the C_i elementary world–volume field is

$$h[C_i] = \frac{1}{3}. \tag{4.4.84}$$

The stability subgroup of the permuted cycles is obtained permuting the indices $1, 2, 3$ in eq. (4.4.83) and we reach the obvious conclusion

$$h[A_i] = h[B_j] = h[C_\ell] = \frac{1}{3}. \tag{4.4.85}$$

This matches with the previous result (4.3.3) on the spectrum of chiral operators, which are predicted of the form

$$\text{chiral operators} = \mathrm{Tr}\,(A_{i_1}\, B_{j_1}\, C_{\ell_1}\, \ldots\, A_{i_k}\, B_{j_k}\, C_{\ell_k}) \tag{4.4.86}$$

and should have conformal weight $E = k$. Indeed, we have $k \times (\frac{1}{3} + \frac{1}{3} + \frac{1}{3}) = k$!

4.5 Conclusions

We saw, using geometrical intuition, that there is a set of fundamental fields which are likely to be the fundamental degrees of freedom of the CFT's corresponding to Q^{111} or M^{111}. The entire KK spectrum and the existence of baryons of given quantum numbers can be explained in terms of them. This fact (expecially the formula (4.4.49)) constitutes in my opinion a strong non–trivial check of the AdS/CFT correspondence.

Candidate three-dimensional gauge theories which should flow in the IR to the superconformal fixed points dual to the AdS_4 compactifications have also been discussed in this thesis. The fundamental fields are the elementary chiral multiplets of these gauge theories.

The main problem which has not been solved is the existence of chiral operators in the gauge theory that have no counterpart in the KK spectrum. These are the non completely flavour symmetric chiral operators. Their existence is due to the fact that, differently from the case of T^{11}, we are not able to write any superpotential of dimension two. If the proposed gauge theories are correct, the dynamical mechanism responsible for the disappearing of the non symmetric operators in the IR has still to be clarified. It is probably of non–perturbative nature.

It would be quite helpful to have a description of the conifold as a deformation of an orbifold singularity [13], [65]. It would provide an holographic description of the RG flow between two different CFT theories and it would also help in checking whether the proposed gauge theories are correct or require to be slightly modified by the introduction of new fields. Another direction of possible improvement of our theory consists in considering the Chern Simons coupling, which we have set to zero.

Appendix A

Conventions for the M^{111} space

The Gell–Mann matrices are:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \tfrac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{A.0.1}$$

The Pauli matrices are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.0.2}$$

The structure constants of $SU(3)$ are given by $f_{ijk} = f_{[ijk]}$, $[\lambda_i, \lambda_j] = 2i f_{ijk} \lambda_k$

$$\begin{aligned} f_{123} &= 1\,, \\ f_{147} &= \tfrac{1}{2}\,, \quad f_{156} = -\tfrac{1}{2}\,, \quad f_{246} = \tfrac{1}{2}\,, f_{257} = \tfrac{1}{2}\,, \quad f_{345} = \tfrac{1}{2}\,, \quad f_{367} = -\tfrac{1}{2}\,, \\ f_{458} &= \tfrac{\sqrt{3}}{2}\,, \quad f_{678} = \tfrac{\sqrt{3}}{2}\,. \end{aligned} \tag{A.0.3}$$

The generators of $G = SU\,(3) \times SU\,(2) \times U\,(1)$ are:

$$\begin{aligned} SU\,(3): &\quad \frac{i}{2}\lambda_1, \ldots, \frac{i}{2}\lambda_8 \\ SU\,(2): &\quad \frac{i}{2}\sigma_1, \ldots, \frac{i}{2}, \sigma_3 \\ U\,(1): &\quad iY\,. \end{aligned}$$

The orthogonal decomposition gives

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{A.0.4}$$

where $\mathbb{H}$ is a subalgebra of $\mathbb{G}$, and $\mathbb{K}$ is a representation of $\mathbb{H}$.

The generators of $H = SU(2) \times U(1)' \times U(1)''$ are:

$$\begin{aligned} SU(2): &\qquad \frac{i}{2}\lambda_{\dot m} = \frac{i}{2}\lambda_1, \dots, \frac{i}{2}\lambda_3 \\ U(1)': &\qquad Z' = \sqrt{3} i\lambda_8 + i\sigma_3 - 4iY \\ U(1)'': &\qquad Z'' = -\frac{\sqrt{3}}{2} i\lambda_8 + \frac{3}{2} i\sigma_3 \end{aligned}$$

so the generators of the orthogonal space $\mathbb{K}$ are

$$\begin{aligned} \frac{i}{2}\lambda_A &= \frac{i}{2}\lambda_4 \dots \frac{i}{2}\lambda_7, \\ \sigma_m &= \frac{i}{2}\sigma_1, \frac{i}{2}\sigma_2 \\ Z &= \frac{\sqrt{3}}{2} i\lambda_8 + \frac{1}{2} i\sigma_3 + iY \,. \end{aligned} \tag{A.0.5}$$

Due to this decomposition we divide the indices into six groups:

$$\begin{gathered} \dot m, \dot n = 1, 2, 3, \\ Y, \\ m, n = 1, 2, \\ 3, \\ A, B, C = 4, 5, 6, 7, \\ 8\,. \end{gathered} \tag{A.0.6}$$

Other indices used in this context are:

$$\begin{aligned} \Sigma, \Lambda: &\qquad \text{indices of the adjoint representation of } G \\ a, b: &\qquad \text{indices of the vector representation of } SO(7) \\ i, j: &\qquad \text{indices of the vector representation of } SU(2)\,. \end{aligned} \tag{A.0.7}$$

Our conventions for the ε tensors are the following:

$$\begin{array}{lll} SU(2) \subset G: & \varepsilon^{mn} & \varepsilon^{12} = -1 \\ SU(3) \subset G: & \varepsilon^{\dot m \dot n \dot r} & \varepsilon^{\dot 1 \dot 2 \dot 3} = 1 \\ SU(2) \subset H: & \varepsilon^{\dot m \dot n} & \varepsilon^{\dot 1 \dot 2} = 1 \\ SO(7)^c: & \varepsilon^{abcdefg} & \varepsilon^{1234567} = -1\,. \end{array} \tag{A.0.8}$$

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